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M337

Complex analysis

Book D

Applications of complex analysis

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Unit D1
Fluid flows

Introduction

This unit differs from the others in the module in that it seeks to show how complex analysis can be used in the mathematical modelling of a physical process, namely the flow of a *fluid* such as water or air.

If you have experience of mathematical modelling, then the approach adopted here will be familiar to you: start with an initial simplification of a real-world situation, then express what is going on mathematically and make progress by means of posing and solving problems. Later we interpret the solutions in physical terms, and conclude by discussing how good the model is and where it could be improved. If you have not done mathematical modelling before, then you need not feel apprehensive about it. This unit primarily focuses on the mathematical aspects of a fluid flow model; however, it will help your appreciation of the content if you are able to think in terms of physical quantities such as *velocity* and *pressure*.

Although the assumptions we make in our mathematical model are quite restrictive, the resulting model is still significant in terms of the physical insights that it provides. More realistic models of fluid flows have been developed, but their additional realism is paid for by an increase in mathematical complexity. The model considered here has the virtue that it is understandable, and it is sufficiently rich that its predictions are illuminating. For example, the model predicts correctly the presence of the upward force that keeps an aeroplane in flight. This model of fluid flow can be developed without using complex analysis; however, the language of complex analysis allows us to demonstrate the natural relationships within this model in a transparent manner.

In Section 1 we set up the mathematical model for fluid flows using a complex-valued function to describe the steady fluid velocity within a cross-sectional plane. It turns out that, under suitable modelling assumptions, the conjugate of this velocity function is analytic, so it can be studied using techniques from complex analysis.

In particular, the conjugate velocity function has a primitive (on any suitable region) known as a *complex potential function*, which we introduce in Section 2. The *streamlines*, that is, the paths of points moving with the fluid, can be described simply in terms of the complex potential function. We look at several examples of simple fluid flows, and derive their complex potential functions and the corresponding streamline patterns.

Section 3 is devoted to the study of an important family of conformal mappings arising from the so-called *Joukowski functions*, which play a key role in analysing fluid flows modelled with complex analysis.

In Section 4 we turn to the investigation of flow patterns in a fluid stream passing around a solid object, or *obstacle*, first covering in detail the case of flow past a circular cylinder. We then show how conformal mappings, and in particular Joukowski functions, can be used to relate the flow past a circular cylinder to the flow past obstacles of other shapes.

In Section 5 this conformal mapping technique is developed further to study flow past an *aerofoil*. The section ends with a brief discussion of the *force* on an obstacle due to the flow past it, and of how well the predictions of the model compare with reality.

Unit guide

Sections 4 and 5 form the heaviest parts of this unit, and you will probably need to spend much of your study time on them. You should find that Sections 2 and 3 are relatively light, although essential in terms of building up your ability to visualise how fluids in motion behave according to the model.

1 Setting up the model

After working through this section, you should be able to:

- appreciate the modelling assumptions that are made in order to represent the velocity of a fluid flow by a continuous complex function
- apply the formula for the *components* of velocity at any point, in a specified direction
- explain what is meant by *stagnation point*, *streamline*, *unit-speed parametrisation*, *circulation*, *flux*, *locally circulation-free*, *locally flux-free*, *ideal flow*, *source*, *sink* and *vortex*
- understand why the conjugate of an ideal flow velocity function is an analytic function, and vice versa
- establish whether a velocity function is locally circulation-free, locally flux-free, or both.

1.1 A complex-valued velocity function

The aim of this unit is to develop a mathematical model that can be applied to the *flow* (or motion) of a *fluid*. We consider a fluid to be a substance that behaves like a liquid or a gas. Both water (a liquid) and air (a gas) are fluids, and your everyday experience of how these substances behave in motion should give you some feeling for what we intend to model.

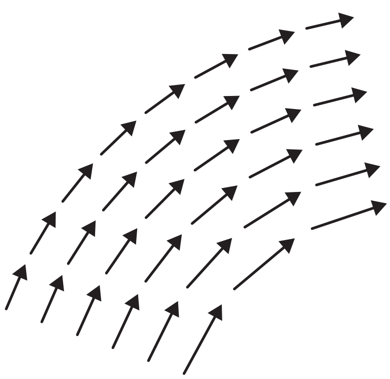


Figure 1.1 Visualising a fluid flow using an arrow diagram

One way of visualising a fluid flow is shown in Figure 1.1. The diagram represents a plane cross-section, at a particular instant in time, of the flow of some fluid such as water.

Each arrow in this *arrow diagram* represents the instantaneous *velocity* vector of the fluid at the point from which the arrow is directed. The magnitude of this velocity (that is, the fluid speed) is represented by the length of the arrow, while the flow direction is given by the direction of the arrow. (We briefly discussed such arrow diagrams at the end of Section 3 of Unit A2, where we referred to them as *vector fields*.)

Photographs of flows similar to Figure 1.1 (but without the arrowheads) can be obtained by the insertion of small neutrally buoyant beads (beads with the same density as that of the fluid) into the flow. By taking a long-exposure photograph over a short interval of time, the movements of the beads can be visualised as streaks on the photograph. Since the beads are carried along by the fluid, the streak lengths and directions indicate fluid velocity, as represented in Figure 1.1 by the arrows.

Suppose that we regard a flow cross-section, such as the one shown in Figure 1.1, as lying in a region of the complex plane. Then each point in the region can be described by a complex number $z = x + iy$. The flow velocity at the point z is characterised completely by its magnitude and direction, so it can be represented by a complex number $q(z)$, where $|q(z)|$ is the fluid speed at z, and $\mathrm{Arg}(q(z))$, the principal argument of $q(z)$, gives the direction of flow at that point. Thus the velocities at all points within the fluid cross-section can be described by a complex function q, whose domain is the region occupied by the fluid. We call q a **velocity function**.

Examples of points z and corresponding values $q(z)$ of the velocity function, for a particular flow, are shown in the arrow diagram in Figure 1.2.

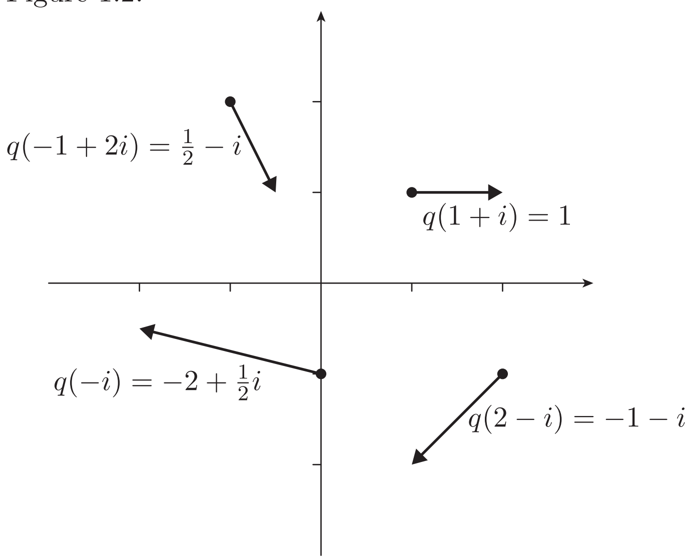


Figure 1.2 The velocity $q(z)$ at various points z

The use of a complex function q to describe a fluid velocity relies on several significant modelling assumptions.

First, Figure 1.1 depicts a cross-section of a flow which must in reality be three-dimensional, although the two-dimensional diagram is intended to show all the significant features of the flow. You should imagine that any point moving with the fluid in the plane shown continues to reside in that plane at all later times. (You can visualise a point moving with the fluid as the centre of a very small neutrally buoyant bead carried along by the flow.) Moreover, any cross-section of the flow that is parallel to the chosen plane gives a velocity function identical to that in the original plane. Such a flow is called *two-dimensional*.

A second modelling assumption becomes apparent when we recall that a real fluid is actually composed of molecules that move and collide with one another, and with solid objects. In our model, however, we have assumed that a fluid is a *continuum*, and we also assume that any property of a fluid, such as velocity, can be defined by a function whose values vary continuously in the region occupied by the fluid. These assumptions are together called the *continuum assumption*.

These first two assumptions describe real fluids accurately, except at microscopic scales.

Our third modelling assumption concerns what happens when the fluid moves on from the pattern at a particular instant, such as that in Figure 1.1. In reality, the fluid velocity at each point will usually change with time, but in our model this complication is avoided by assuming that the flow is *steady*; this means that, for each point in the region occupied by the fluid, the velocity of the fluid at that point is independent of time. Thus Figure 1.1 represents the flow pattern at *any* time.

Note that the assumption of steady flow is *not* the same as restricting the velocity of a point moving with the fluid to be constant as time passes. What is fixed in time is the flow velocity observed at any specified point of the flow region.

We now summarise the basic features of our model for fluid flows, which allows us to investigate such flows using complex analysis.

Basic fluid flow model

We assume that

- the flow is two-dimensional
- the fluid forms a continuum, and any variation of the flow velocity within this continuum is continuous
- the flow is steady.

With these assumptions, we can represent the flow velocity at all times by a continuous complex function q, whose domain is the region occupied by the fluid.

The requirements that the flow be both two-dimensional and steady can seem restrictive, since many possible flows are excluded. However, there are significant types of flow that satisfy these conditions, at least approximately; for example, the motion of air past an aeroplane wing in flight can for many purposes be modelled satisfactorily by a two-dimensional steady flow.

At a point z where the fluid is at rest (has zero speed), the flow velocity satisfies $q(z) = 0$. Such a point is called a **stagnation point** of the flow.

Example 1.1

Sketch the arrow diagram for the flow with velocity function

$$q(z) = e^{i\pi/4} \quad (z \in \mathbb{C}).$$

Solution

For each z, the flow has speed $|e^{i\pi/4}| = 1$ and direction given by $\operatorname{Arg}(e^{i\pi/4}) = \pi/4$. The arrow diagram is therefore as shown in Figure 1.3.

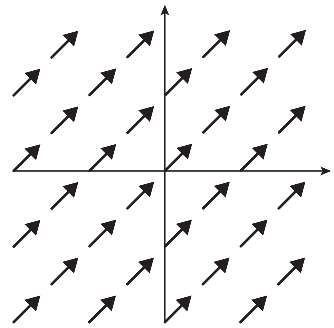


Figure 1.3 Arrow diagram for the flow with velocity function $q(z) = e^{i\pi/4}$

The velocity function in Example 1.1 is a constant function. A flow given by a constant velocity function is called a **uniform flow** or **uniform stream**.

Figure 1.3 was easy to draw because all the arrows were the same length. When the arrow lengths (representing flow speeds) vary widely, it may not be a straightforward matter to sketch this type of picture. For this reason such diagrams are often drawn with arrows of fixed length, representing the direction of the flow at each point but not variations in speed. You are asked to adopt this approach in the next exercise.

Exercise 1.1

Sketch the arrow diagram for each of the following velocity functions, using arrows of fixed length.

(a) $q(z) = z \quad (z \in \mathbb{C})$ (b) $q(z) = i/\overline{z} \quad (z \in \mathbb{C} - \{0\})$

Determining the velocity function is one of the most important parts of understanding the behaviour of a fluid flow that satisfies the basic fluid flow model. As you will see, the velocity function can then be used to obtain information about the actual motion of the fluid.

The trajectory of a point moving with the fluid forms a path

$$\Gamma : \gamma(t) \quad (t \in I),$$

which is called a *streamline*. In this context the parameter t represents time. At any point $\gamma(t)$ on the streamline at time t, the velocity is $q(\gamma(t))$, and this is equal to the rate of change of position $\gamma'(t)$, which is a tangent vector to the streamline. (The interpretation of $\gamma'(t)$ as the tangent vector to the path $\Gamma : \gamma(t)$ $(t \in I)$ at the point $\gamma(t)$ was introduced in Subsection 4.1 of Unit A4, as was the term *smooth path*.)

We make the following definitions.

Definitions

A **streamline** (or **flow line**) through the point z_0, for a flow with velocity function q, is a smooth path $\Gamma : \gamma(t)$ $(t \in I)$ such that

- $\gamma'(t) = q(\gamma(t)), \quad \text{for } t \in I$
- $z_0 = \gamma(t_0), \quad \text{for some } t_0 \in I.$

If $q(z_0) = 0$ (that is, if z_0 is a stagnation point), then the point z_0 is a **degenerate streamline**, with constant parametrisation

$$\gamma(t) = z_0 \quad (t \in I).$$

A diagram showing a collection of streamlines for a flow is called a **streamline diagram**.

Remarks

1. Any streamline $\Gamma : \gamma$ $(t \in I)$ through the point z_0 has many parametrisations that differ from γ only by a time translation (that is, by having the image z_0 at a different value of t_0). This corresponds to the fact that, for a steady flow, a description of the flow pattern is unaffected by the moment at which you choose to set a clock to measure time.
2. A degenerate streamline is considered to be a special type of streamline.
3. Note that streamlines may or may not be straight lines.
4. Some texts refer to streamline diagrams as *phase portraits*.

Consider, for example, the flow velocity function $q(z) = e^{i\pi/4}$ $(z \in \mathbb{C})$ from Example 1.1. In this case we have

$$\gamma'(t) = q(\gamma(t)) = e^{i\pi/4}, \quad \text{for } t \in \mathbb{R},$$

so a streamline through the point z_0 is given by the parametrisation

$$\gamma(t) = z_0 + te^{i\pi/4}, \quad \text{for } t \in \mathbb{R},$$

where we have taken $t_0 = 0$.

To return to the general situation, note that we can use the condition $\gamma'(t) = q(\gamma(t))$ to sketch streamlines. At each point z, the velocity $q(z)$ is a tangent vector to a streamline which passes through z (unless z is a stagnation point). For example, we constructed the streamline diagram in Figure 1.4(a) by 'joining up the arrows' of the arrow diagram in Figure 1.1. We indicate the flow direction along each streamline by a single arrowhead. Similarly, the streamline diagram for the flow of Example 1.1 is shown in Figure 1.4(b).

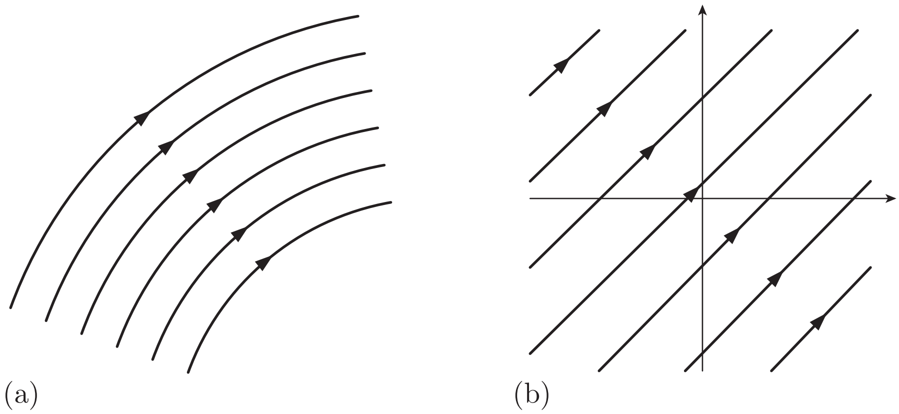


Figure 1.4 (a) Streamlines with directions of flow indicated
(b) Streamlines for the flow with velocity function $q(z) = e^{i\pi/4}$

The next exercise asks you to sketch streamlines for the flows that you considered in Exercise 1.1.

Exercise 1.2

Sketch a few streamlines for each of the flows with the following velocity functions.

(a) $q(z) = z \quad (z \in \mathbb{C})$ (b) $q(z) = i/\overline{z} \quad (z \in \mathbb{C} - \{0\})$

1.2 Developing the model

We have chosen to restrict our attention to the steady, two-dimensional flow of a fluid forming a continuum, for which the fluid velocity throughout its domain is represented by a continuous complex function q. We now make two further modelling assumptions in order to represent the physical situation with a greater degree of accuracy, and to make more specific the mathematical problem of finding the flow velocity function q for a given flow.

In order to describe these restrictions, we need to introduce the ideas of *circulation* and *flux*. As a necessary preliminary to this, we derive a useful expression for the *component* of velocity in a given direction and introduce a particularly convenient type of parametrisation.

Suppose that the given direction is at an angle θ from the positive real axis, specified by the complex number $e^{i\theta}$. Let $q(z)$ be the complex number representing the velocity at a point z. Thinking of $q(z)$ as a vector, we let $q_\theta(z)$ denote the component of $q(z)$ in the direction specified by $e^{i\theta}$. Figure 1.5 shows two cases of how $q_\theta(z)$ is obtained by projecting $q(z)$ perpendicularly onto the line through 0 and $e^{i\theta}$.

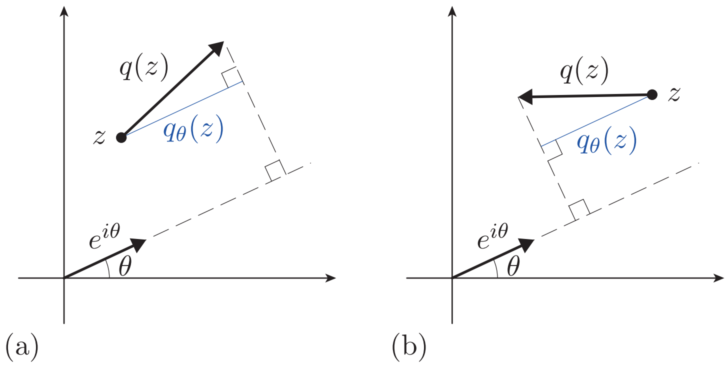


Figure 1.5 Component $q_\theta(z)$ of $q(z)$ in the direction $e^{i\theta}$: (a) $q_\theta(z) > 0$, (b) $q_\theta(z) < 0$

Note that the component $q_\theta(z)$ is positive, negative or zero, depending on whether the projection of the vector $q(z)$ onto the line through 0 and $e^{i\theta}$ points in the same direction as $e^{i\theta}$, points in the opposite direction to $e^{i\theta}$, or is a single point.

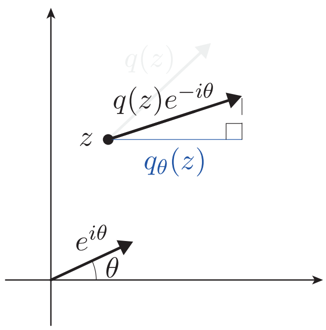


Figure 1.6 Rotating $q(z)$ to calculate the component $q_\theta(z)$ in Figure 1.5(a)

We now seek an expression for the component $q_\theta(z)$ in terms of $q(z)$ and θ. If we rotate both the complex number $q(z)$ and the θ-direction line about z through the angle $-\theta$, then $q(z)$ goes to $q(z)e^{-i\theta}$ and the θ-direction line becomes parallel to the real axis. This is shown in Figure 1.6 for the situation in Figure 1.5(a). The component of $q(z)e^{-i\theta}$ in the positive x-direction is

$$\operatorname{Re}(q(z)e^{-i\theta}),$$

and this is the same number as the component $q_\theta(z)$. Hence

$$q_\theta(z) = \operatorname{Re}(q(z)e^{-i\theta}).$$

This last equation gives an expression for the component $q_\theta(z)$ of $q(z)$ in the direction specified by $e^{i\theta}$. It turns out that the following variation of this formula is more useful:

$$q_\theta(z) = \operatorname{Re}(q(z)e^{-i\theta}) = \operatorname{Re}(\overline{q(z)e^{-i\theta}}) = \operatorname{Re}(\overline{q(z)}e^{i\theta}). \tag{1.1}$$

This holds because the real parts of a complex number and its conjugate are the same.

Example 1.2

Find the component of $q(z) = 2i$ ($z \in \mathbb{C}$) in the direction specified by $e^{i\pi/6}$.

Solution

From equation (1.1), with $q(z) = 2i$ and $\theta = \pi/6$, we have

$$\begin{aligned} q_{\pi/6}(z) &= \operatorname{Re}(\overline{2i}\, e^{i\pi/6}) \\ &= \operatorname{Re}\left(-2i\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right) \\ &= 2\sin\frac{\pi}{6} = 1. \end{aligned}$$

Exercise 1.3

Find the component of $q(z) = 5e^{-i\pi/6}$ $(z \in \mathbb{C})$ in the direction specified by $e^{2i\pi/3}$.

Exercise 1.4

Use equation (1.1) to show that the component of the velocity $q(z)$ in the direction specified by the angle $\theta - \pi/2$ can be expressed as

$$q_{(\theta-\pi/2)}(z) = \operatorname{Im}(\overline{q(z)}e^{i\theta}).$$

Equation (1.1) and the result of Exercise 1.4 can be rewritten in terms of the **conjugate velocity function** $\overline{q}$, which has the same domain as q and has rule

$$\overline{q}(z) = \overline{q(z)}.$$

We have

$$q_\theta(z) = \operatorname{Re}(\overline{q}(z)e^{i\theta}) \quad \text{and} \quad q_{(\theta-\pi/2)}(z) = \operatorname{Im}(\overline{q}(z)e^{i\theta}), \tag{1.2}$$

which will be of use shortly.

Now suppose that $\Gamma : \gamma(t)$ $(t \in I)$ is any smooth path of finite length. It will be convenient to assume that γ is a **unit-speed parametrisation**; that is,

$$|\gamma'(t)| = 1, \quad \text{for } t \in I.$$

For example, $\gamma(t) = 2\left(\cos \frac{1}{2}t + i \sin \frac{1}{2}t\right)$ $(t \in [0, 4\pi])$ is a unit-speed parametrisation of the circle $|z| = 2$.

The following result shows that we can find a unit-speed parametrisation of any smooth path of finite length.

Theorem 1.1

Let $\Gamma : \gamma_1(t)$ $(t \in [a, b])$ be a smooth path of length L. Then there is another smooth parametrisation $\gamma(s)$ $(s \in [0, L])$ of Γ such that

$$|\gamma'(s)| = 1, \quad \text{for } 0 \le s \le L.$$

Proof Since γ_1 is smooth, the real function $t \longmapsto |\gamma_1'(t)|$ is continuous on $[a, b]$, so it has a primitive h, say. Assuming, as we can, that $h(a) = 0$, we obtain

$$h(b) = h(b) - h(a) = \int_a^b |\gamma_1'(u)|\, du = L,$$

by the Fundamental Theorem of Calculus (Theorem 1.2 of Unit B1).

Since $h'(t) = |\gamma_1'(t)| > 0$, we see from the Inverse Function Rule (Theorem 3.2 of Unit A4) that $\left(h^{-1}\right)'(s) = 1/h'(t) > 0$, for $s \in [0, L]$, where $t = h^{-1}(s)$. It follows that the function

$$\gamma(s) = \gamma_1\left(h^{-1}(s)\right) \quad (s \in [0, L])$$

satisfies

$$|\gamma'(s)| = |\gamma_1'\left(h^{-1}(s)\right)\left(h^{-1}\right)'(s)| = \frac{|\gamma_1'(t)|}{h'(t)} = 1, \quad \text{for } 0 \leq s \leq L,$$

by the Chain Rule (Theorem 3.1 of Unit A4).

Hence γ is a smooth unit-speed parametrisation of Γ, as required. ■

Remark

If $|\gamma'(s)| = 1$ for $0 \leq s \leq L$, then the parameter s measures length along Γ since if $0 \leq s_1 < s_2 \leq L$, then the length of Γ from $\gamma(s_1)$ to $\gamma(s_2)$ is

$$\int_{s_1}^{s_2} |\gamma'(s)|\, ds = \int_{s_1}^{s_2} 1\, ds = s_2 - s_1.$$

Note that it is traditional to use the letter s as a parameter for unit-speed parametrisations, since s represents length.

Definitions

Let $\Gamma : \gamma(s)$ $(s \in [0, L])$ be a smooth path with unit-speed parametrisation which lies in the domain of a flow with velocity function q. Then, for each $s \in [0, L]$, the velocity $q(\gamma(s))$ has

- **tangential component** $q_T(s)$ in the direction specified by $\gamma'(s)$
- **normal component** $q_N(s)$ in the direction specified by $-i\gamma'(s)$.

These components are shown in Figure 1.7 in a case when both are positive. Since $|\gamma'(s)| = 1$, it follows that $\gamma'(s)$ is a *unit* tangent vector at the point $\gamma(s)$ on Γ, for each $s \in [0, L]$, and $-i\gamma'(s)$ is a unit normal vector.

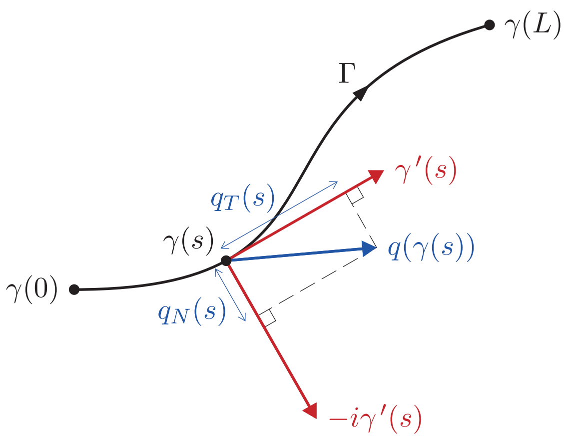


Figure 1.7 Components of q that are tangential and normal to a smooth path Γ

We now define the important concepts of *circulation* and *flux* in terms of these velocity components. Circulation and flux are both real quantities associated with a smooth path and a flow velocity, which have physical interpretations related to the fluid flow.

Definitions

Let $\Gamma : \gamma(s)$ $(s \in [0, L])$ be a smooth path with unit-speed parametrisation which lies in the domain of a flow with continuous velocity function q.

- The **circulation** of q along Γ is
$$\mathcal{C}_\Gamma = \int_0^L q_T(s)\, ds.$$
- The **flux** of q across Γ is
$$\mathcal{F}_\Gamma = \int_0^L q_N(s)\, ds.$$

In words, the circulation of q along Γ is

$$L \times \text{the average value of } q_T \text{ on } \Gamma,$$

and the flux of q across Γ is

$$L \times \text{the average value of } q_N \text{ on } \Gamma.$$

Roughly speaking, circulation measures the overall rate of the fluid flow *along* the path Γ, and flux measures the overall rate at which fluid flows *across* Γ.

Now, for each $s \in [0, L]$, $\gamma'(s)$ has modulus 1, so we have $\gamma'(s) = e^{i\theta}$ for some θ. Thus

$$q_T(s) = q_\theta(\gamma(s)) \quad \text{and} \quad q_N(s) = q_{(\theta-\pi/2)}(\gamma(s)),$$

so, by equations (1.2),

$$q_T(s) = \operatorname{Re}(\overline{q}(\gamma(s))\,\gamma'(s)) \quad \text{and} \quad q_N(s) = \operatorname{Im}(\overline{q}(\gamma(s))\,\gamma'(s)). \tag{1.3}$$

It follows from equations (1.3), and the definitions of circulation and flux, that

$$\begin{aligned}\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma &= \int_0^L (q_T(s) + iq_N(s))\, ds \\ &= \int_0^L \overline{q}(\gamma(s))\,\gamma'(s)\, ds \\ &= \int_\Gamma \overline{q}(z)\, dz.\end{aligned}$$

This final formula is actually valid for any contour Γ, provided that the circulation $\mathcal{C}_\Gamma$ is defined as the sum of the circulations along the smooth paths which form Γ, and the flux $\mathcal{F}_\Gamma$ is defined as the sum of the fluxes across these smooth paths.

Hence we obtain the following useful formula.

Theorem 1.2 Circulation and Flux Contour Integral

For any contour Γ in the domain of a flow with continuous velocity function q, we have

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \overline{q}(z)\,dz.$$

We are now almost ready to make our final modelling assumptions. These are based on the following definitions.

Definitions

A flow with continuous velocity function q and domain a region $\mathcal{R}$ is

- **locally circulation-free** if $\mathcal{C}_\Gamma = 0$ for each simple-closed contour Γ in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$
- **locally flux-free** if $\mathcal{F}_\Gamma = 0$ for each simple-closed contour Γ in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$.

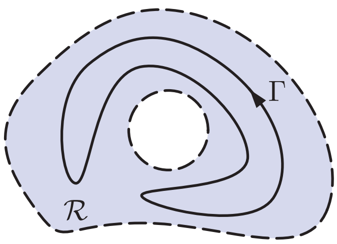


Figure 1.8 A simple-closed contour Γ in a region $\mathcal{R}$

Note that if $\mathcal{R}$ is simply connected, then the inside of any simple-closed contour in $\mathcal{R}$ automatically lies in $\mathcal{R}$. Figure 1.8 shows a region $\mathcal{R}$ that is not simply connected and a simple-closed contour Γ in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$.

We will assume for our model that any flow we consider has a velocity function that satisfies both of the conditions just defined.

Definition

An **ideal flow** is a fluid flow, defined by a continuous velocity function on a region, that is locally circulation-free and locally flux-free.

The locally flux-free condition is equivalent, for a constant-density fluid, to the *principle of conservation of mass*, which states that mass (matter) is neither created nor destroyed. In the context of fluid flows this means the following: for any simple-closed contour Γ whose inside is in the flow region $\mathcal{R}$, any flow of fluid mass inwards across Γ must be exactly balanced by a flow of fluid mass outwards over the same period of time, so the mass of fluid associated with the inside of Γ remains constant.

The need for the locally circulation-free condition is less obvious physically, and it is in fact a simplifying assumption. Its effect is that any small neutrally buoyant bead, whose surface velocity matches that of the adjacent fluid, *does not rotate*. For this reason, a flow with this property is also called *irrotational*.

1.3 The conjugate velocity function

An ideal flow is described by a continuous velocity function q, with domain a region $\mathcal{R}$. As pointed out in the previous subsection, the conjugate velocity function $\overline{q}$ is defined by

$$\overline{q}(z) = \overline{q(z)} \quad (z \in \mathcal{R}).$$

The function $\overline{q}$ is continuous, since q is continuous.

Now, by the definition of an ideal flow, q is locally both circulation-free and flux-free. Hence, by Theorem 1.2, we have

$$\int_\Gamma \overline{q}(z)\,dz = 0, \tag{1.4}$$

for each simple-closed contour Γ in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$. In particular, equation (1.4) holds for all rectangular contours whose insides lie in $\mathcal{R}$, so it follows from Morera's Theorem (Theorem 5.4 of Unit B2) that the conjugate velocity function $\overline{q}$ is analytic on $\mathcal{R}$.

The converse of this result also holds. If $\overline{q}$ is analytic on a region $\mathcal{R}$, then, by Cauchy's Theorem (Theorem 1.2 of Unit B2),

$$\int_\Gamma \overline{q}(z)\,dz = 0,$$

for each simple-closed contour Γ in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$. Therefore, by Theorem 1.2, the flow with velocity function q is locally both circulation-free and flux-free. We have thus established the following result.

Theorem 1.3

A steady two-dimensional fluid flow with continuous velocity function q on a region $\mathcal{R}$ is an ideal flow if and only if its conjugate velocity function $\overline{q}$ is analytic on $\mathcal{R}$.

This theorem provides a method for finding a profusion of ideal flows, since the conjugate of each analytic function is the velocity function for an ideal flow.

Example 1.3

Find the ideal flow velocity function corresponding to the analytic function

$$g(z) = e^{-i\pi/4} \quad (z \in \mathbb{C}).$$

Solution

The conjugate velocity function is $\overline{q}(z) = e^{-i\pi/4}$, so the ideal flow velocity function is

$$q(z) = e^{i\pi/4} \quad (z \in \mathbb{C}).$$

Here are some exercises that relate to Theorem 1.3.

Exercise 1.5

For each of the following analytic functions, find the corresponding ideal flow velocity function. Draw a fixed-length arrow diagram to illustrate each answer.

(a) $g(z) = z \quad (z \in \mathbb{C})$ (b) $g(z) = 1/z \quad (z \in \mathbb{C} - \{0\})$

(c) $g(z) = i/z \quad (z \in \mathbb{C} - \{0\})$

Exercise 1.6

Determine which of the following velocity functions defines an ideal flow.

(a) $q(z) = z \quad (z \in \mathbb{C})$ (b) $q(z) = i/\overline{z} \quad (z \in \mathbb{C} - \{0\})$

Exercise 1.7

Let Γ be the unit circle $\{z : |z| = 1\}$.

(a) Use Theorem 1.2 to find the flux across Γ of the velocity function $q(z) = 1/\overline{z}$ $(z \in \mathbb{C} - \{0\})$.

(b) Use Theorem 1.2 to find the circulation along Γ of the velocity function $q(z) = -i/\overline{z}$ $(z \in \mathbb{C} - \{0\})$.

(c) Explain why your answers to parts (a) and (b) do not contradict Theorem 1.3.

Exercise 1.7 demonstrates that the flux or circulation of a flow can be non-zero for certain closed contours even when the flow is *locally* both circulation-free and flux-free. The example of Exercise 1.5(b) corresponds to outward flow of fluid along rays from the origin, a point where the fluid is somehow appearing; this flow is described as a *source* of strength $\mathcal{F}_\Gamma = 2\pi$ at the origin. The example of Exercise 1.5(c) corresponds to clockwise flow of fluid on circles around the origin; this flow is described as a *vortex* of strength $|\mathcal{C}_\Gamma| = 2\pi$ at the origin. More generally, we have the following definitions, which are illustrated by Figure 1.9.

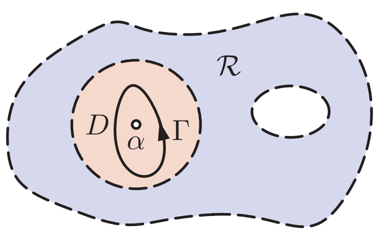


Figure 1.9 The geometric context for α to be a source, a sink or a vortex

Definitions

Let q be a velocity function for an ideal flow with flow region $\mathcal{R}$, and let D be a punctured open disc in $\mathcal{R}$ with centre α. Then

- α is a **source** of strength $\mathcal{F}$ if $\mathcal{F}_\Gamma = \mathcal{F} > 0$ for each simple-closed contour Γ in D that surrounds α
- α is a **sink** of strength $|\mathcal{F}|$ if $\mathcal{F}_\Gamma = \mathcal{F} < 0$ for each simple-closed contour Γ in D that surrounds α
- α is a **vortex** of strength $|\mathcal{C}|$ if $\mathcal{C}_\Gamma = \mathcal{C} \neq 0$ for each simple-closed contour Γ in D that surrounds α.

  An **anticlockwise vortex** is a vortex with $\mathcal{C} > 0$, and a **clockwise vortex** is a vortex with $\mathcal{C} < 0$.

Remarks

1. If the point α is a source, a sink or a vortex, then it is not in the region $\mathcal{R}$. In this case $\overline{q}$ has an isolated singularity at α (see Subsection 1.1 of Unit B4).
2. The fact that $\mathcal{F}_\Gamma$ and $\mathcal{C}_\Gamma$ are independent of the choice of the simple-closed contour Γ is a consequence of the Shrinking Contour Theorem (Theorem 1.4 of Unit B2).

To conclude this section we ask you to use the Cauchy–Riemann equations (Subsection 2.1 of Unit A4) to obtain an alternative formulation of the condition that a velocity function for an ideal flow should be locally both circulation-free and flux-free.

Exercise 1.8

(a) Let q be a continuous velocity function with domain a region $\mathcal{R}$, and suppose that $q_1 = \operatorname{Re} q$ and $q_2 = \operatorname{Im} q$ have partial derivatives with respect to x and y that are continuous on $\mathcal{R}$.

Prove that q is a velocity function for an ideal flow on $\mathcal{R}$ if and only if

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0 \quad \text{and} \quad \frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0 \quad \text{on } \mathcal{R}.$$

(b) Use your result from part (a) to verify that the velocity function

$$q(z) = i/\overline{z} \quad (z \in \mathbb{C} - \{0\})$$

represents an ideal flow.

Remark

The two equations in Exercise 1.8(a) can be used to show that if q is a velocity function for an ideal flow, then

$$\frac{\partial^2 q_1}{\partial x^2} + \frac{\partial^2 q_1}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 q_2}{\partial x^2} + \frac{\partial^2 q_2}{\partial y^2} = 0 \quad \text{on } \mathcal{R}.$$

Thus both q_1 and q_2 satisfy the second-order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

You met this equation in Subsection 2.2 of Unit A4. It is called *Laplace's equation* and it appears in many other branches of mathematics and physics, such as electrostatics. Any function that is a solution of Laplace's equation is called a **harmonic function**. Thus the real and imaginary parts of the velocity function for any ideal flow are harmonic functions, as are the real and imaginary parts of any analytic function. It is known that solutions of Laplace's equation, and all their partial derivatives of all orders, are continuous functions, except possibly at boundary points of the regions where they are defined, and this is a desirable physical property for modelling fluids.

The solution to Exercise 1.8(a) showed that a function q is the velocity function of an ideal flow (that is, it is locally both circulation-free and flux-free) if and only if the two equations given in that exercise both hold. This result can be separated into two constituent parts as follows. We will not provide a proof for this theorem.

Theorem 1.4

Let q be a continuous velocity function on a region $\mathcal{R}$, and suppose that $q_1 = \operatorname{Re} q$ and $q_2 = \operatorname{Im} q$ have partial derivatives with respect to x and y that are continuous on $\mathcal{R}$.

The flow with velocity function q is

(a) locally circulation-free if and only if

$$\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0 \quad \text{on } \mathcal{R}$$

(b) locally flux-free if and only if

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0 \quad \text{on } \mathcal{R}.$$

These conditions on q_1 and q_2 are often easier to apply than the original definitions of locally flux-free and locally circulation-free, especially when only one of these properties holds.

Exercise 1.9

Use Theorem 1.4 to show that

(a) the flow with velocity function $q(z) = z$ $(z \in \mathbb{C})$ is locally circulation-free, but is not locally flux-free

(b) the flow with velocity function $q(z) = iz$ $(z \in \mathbb{C})$ is locally flux-free, but is not locally circulation-free.

Remark

In vector calculus notation, the locally circulation-free and locally flux-free conditions in Theorem 1.4 are expressed as $\operatorname{curl} \mathbf{q} = \mathbf{0}$ and $\operatorname{div} \mathbf{q} = 0$, respectively.

In the next section you will see further consequences of the close link between ideal flows and analytic functions.

Further exercises

Exercise 1.10

Find the component of $q(z) = 3e^{7i\pi/12}$ ($z \in \mathbb{C}$) in the direction specified by each of the following complex numbers.

(a) $e^{-2i\pi/3}$ (b) $-ie^{-2i\pi/3}$

Exercise 1.11

Use equations (1.2) to show that if θ is any angle, then a velocity function q can be expressed in terms of its components by

$$q(z) = (q_\theta(z) - iq_{(\theta-\pi/2)}(z))e^{i\theta}.$$

Exercise 1.12

Consider the velocity function

$$q(z) = 1 - \frac{1}{\overline{z}^2} \quad (z \in \mathbb{C} - \{0\}).$$

(a) Let Γ be the unit circle $\{z : |z| = 1\}$. Use Theorem 1.2 to show that q has zero circulation along Γ and zero flux across Γ.

(b) Show that q is a velocity function for an ideal flow, using in turn:

(i) Theorem 1.3, (ii) Theorem 1.4.

2 Complex potential functions

After working through this section, you should be able to:

- understand and apply the definition of a *complex potential function*
- find families of streamlines using a complex potential function and *stream function*
- recognise examples of complex potential functions for a flow with a source, sink or vortex
- appreciate the main features of a flow involving a *doublet*, and the effect of placing a doublet in a uniform flow.

2.1 An equation for streamlines

For the rest of this unit we deal only with ideal flows, for which Theorem 1.3 provides a fruitful connection to the theory of analytic functions. You have already seen how to use this connection both to generate velocity functions for ideal flows and to test whether a given velocity function does indeed describe an ideal flow.

We look next at an application to fluid flows of another important result for analytic functions, namely the Primitive Theorem (Theorem 5.3 of Unit B2), which states that if a function is analytic on a simply connected region $\mathcal{R}$, then it has a primitive on $\mathcal{R}$.

Suppose that q is a velocity function for an ideal flow with domain a region $\mathcal{R}$. Then, by Theorem 1.3, the conjugate velocity function $\overline{q}$ is an analytic function with the same domain $\mathcal{R}$. Now suppose that $\overline{q}$ has a primitive Ω, say, on a region $\mathcal{S} \subseteq \mathcal{R}$; that is,

$$\Omega'(z) = \overline{q}(z), \quad \text{for } z \in \mathcal{S}.$$

In the context of ideal flows, a function Ω that is a primitive of $\overline{q}$ is called a **complex potential function**, or simply a **complex potential**, for the flow. By taking the complex conjugate of the equation above, we see that the flow velocity function q is given in terms of the complex potential function by

$$q(z) = \overline{\Omega'(z)}, \quad \text{for } z \in \mathcal{S}.$$

The significance of the complex potential function requires some explanation. First, by Theorem 1.2, we have

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \overline{q}(z)\,dz,$$

where $\mathcal{C}_\Gamma$ and $\mathcal{F}_\Gamma$ are, respectively, the circulation along and the flux across any contour Γ in $\mathcal{R}$.

Suppose that Γ is of finite length and lies within the region $\mathcal{S}$ on which the complex potential Ω is defined. Then we can apply the Fundamental Theorem of Calculus (Theorem 3.1 of Unit B1) to deduce that

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \Omega'(z)\,dz = \Omega(\beta) - \Omega(\alpha),$$

where α and β are the initial and final points of Γ. The real and imaginary parts of this equation are, respectively,

$$\mathcal{C}_\Gamma = \operatorname{Re}\Omega(\beta) - \operatorname{Re}\Omega(\alpha) \quad \text{and} \quad \mathcal{F}_\Gamma = \operatorname{Im}\Omega(\beta) - \operatorname{Im}\Omega(\alpha). \tag{2.1}$$

Now, flux was defined in Subsection 1.2 as the integral along a smooth path $\Gamma : \gamma(s)$ $(s \in I)$ of the normal component $q_N(s)$ of the flow velocity $q(\gamma(s))$. Thus if Γ is a streamline for the flow, that is, a trajectory of points moving with the fluid, then the flow velocity at each point $\gamma(s)$ of Γ is a tangent vector to the path, as shown in Figure 2.1, so $q_N(s) = 0$ on Γ, and hence there is *zero* flux across Γ.

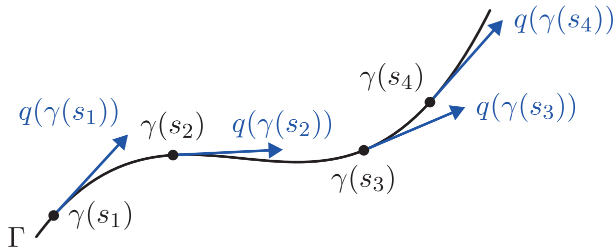


Figure 2.1 Velocity vectors at four points on a streamline Γ

It follows that if α and β both lie on the streamline Γ, then, by applying equations (2.1) to the part of Γ which joins α to β, we obtain

$$\operatorname{Im}\Omega(\beta) = \operatorname{Im}\Omega(\alpha), \tag{2.2}$$

and hence $\operatorname{Im}\Omega(z)$ is constant along the streamline.

On the other hand, if Γ lies in $\mathcal{S}$ and has a smooth unit-speed parametrisation $\gamma(s)$, $0 \le s \le L$, and equation (2.2) holds for all points $\alpha = \gamma(a)$ and $\beta = \gamma(b)$, where $0 \le a < b \le L$, then

$$\int_a^b q_N(s)\,ds = 0, \quad \text{for all } 0 \le a < b \le L,$$

by the definition of flux. Since q_N is a continuous function it follows that $q_N(s) = 0$, for all $s \in [0, L]$, so Γ is part of a streamline for the flow.

Thus we have the following result.

Theorem 2.1

Suppose that an ideal flow is defined on a region $\mathcal{R}$, and Ω is a complex potential function for this flow on a region $\mathcal{S} \subseteq \mathcal{R}$.

Then the streamlines for the flow within $\mathcal{S}$ are the smooth paths with equations of the form $\operatorname{Im}\Omega(z) = k$, for some real constant k.

Remarks

1. This theorem states that for each streamline Γ in $\mathcal{S}$, there is a real constant k such that $\operatorname{Im}\Omega(z) = k$ for all $z \in \Gamma$. However, the set $\{z \in \mathcal{S} : \operatorname{Im}\Omega(z) = k\}$ can comprise more than one streamline. You will see examples of this type in Subsection 2.2.
2. In some cases a velocity function q for an ideal flow with flow region $\mathcal{R}$ may not have a complex potential function on the whole of $\mathcal{R}$, as $\overline{q}$ may not have a primitive on $\mathcal{R}$. However, $\overline{q}$ will have a primitive on any simply connected region $\mathcal{S} \subseteq \mathcal{R}$, by the Primitive Theorem.

   If Γ is a streamline for the flow in such a region $\mathcal{S}$, then we may be able to extend the streamline to other parts of $\mathcal{R}$ by using a sequence of overlapping simply connected regions $\mathcal{S}_n$, for $n = 1, 2, 3, \ldots$, with $\mathcal{S}_1 = \mathcal{S}$, on each of which there is an associated complex potential Ω_n, as illustrated in Figure 2.2.

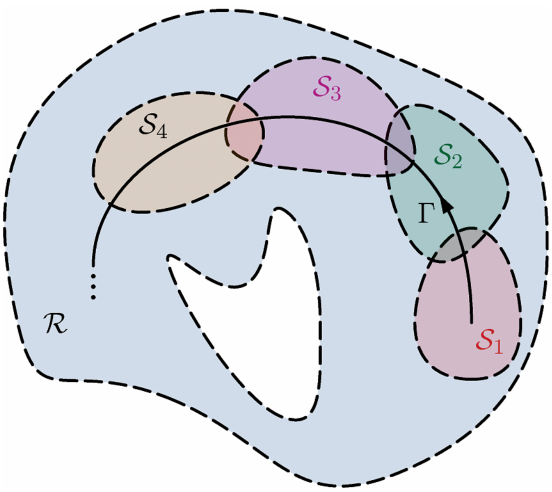


Figure 2.2 Analytic continuation of a complex potential function

Each complex potential of the sequence after the first is related to the previous one by direct analytic continuation (see Section 5 of Unit C1), and the extended streamline Γ is formed from the paths with equations $\operatorname{Im}\Omega_n = k$, with the same value of the constant k for each n.

3. Theorem 2.1 can also be used to show that each point of a flow region has just one streamline through it; we omit the details. Despite this, it can sometimes *appear* that there are two streamlines passing through a point. However, this impression can occur only at a stagnation point, which is itself a degenerate streamline consisting of a single point.

Theorem 2.1 demonstrates one reason why complex potential functions are of interest. For a complex potential function Ω, we write

$$\Omega(z) = \Phi(z) + i\Psi(z),$$

where $\Phi = \operatorname{Re}\Omega$ and $\Psi = \operatorname{Im}\Omega$ are real-valued functions. Then the paths described by the equations of the form

$$\Psi(z) = \text{constant}$$

form the streamlines for an ideal flow. The function $\Psi = \operatorname{Im}\Omega$ is therefore called a **stream function** for the flow.

In the next subsection we look at several ideal flows given by various velocity functions. In each case we find a complex potential function and then the corresponding stream function.

Exercise 2.1

Show that the velocity function q for an ideal flow can be expressed directly in terms of a corresponding stream function Ψ as

$$q(x+iy) = \frac{\partial\Psi}{\partial y}(x,y) - i\frac{\partial\Psi}{\partial x}(x,y).$$

2.2 Simple fluid flows

In this subsection you will see several examples of ideal flows, some of which were mentioned in Section 1. In each case the starting point is a fairly simple velocity function for an ideal flow, from which we derive a complex potential and then obtain the corresponding stream function. This enables the streamline diagram to be drawn. Despite the simplicity of these cases, you will see later that they can be used as the building blocks for more complicated ideal flows.

Our first example is the velocity function for a uniform flow, introduced in Subsection 1.1, which takes the same value at all points.

Uniform flow

The ideal flow with constant velocity function

$$q(z) = \alpha \quad (z \in \mathbb{C}),$$

where $\alpha \in \mathbb{C}$, is a uniform flow.

If $\alpha = 0$, then the fluid velocity is zero everywhere, and nothing moves. If α is non-zero, then the constant velocity implies that throughout the plane, the fluid moves uniformly in the same direction at the same speed.

You can probably see immediately that the streamlines for this flow must be straight lines in the direction of the complex number α, as in Figure 2.3. However, in order to practise using Theorem 2.1, let us derive these streamlines by using a complex potential.

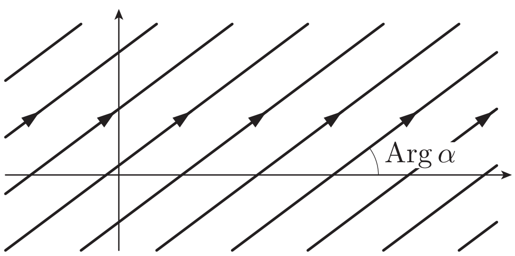


Figure 2.3 Streamlines for a uniform flow

Recall that if q is a velocity function for an ideal flow, then its conjugate function $\overline{q}$ is an analytic function, and any primitive of $\overline{q}$ is a complex potential Ω for the flow. Here we have

$$\overline{q}(z) = \overline{\alpha},$$

so $\Omega(z) = \overline{\alpha}z$ is a complex potential for this flow. Now, by Theorem 2.1, the streamlines for this flow have equations of the form

$$\Psi(z) = \operatorname{Im}\Omega(z) = \operatorname{Im}(\overline{\alpha}z) = \text{constant}.$$

The imaginary part of Ω is the stream function for this flow.

To find the stream function, we let $z = x + iy$ and $\alpha = a + ib$. Then

$$\overline{\alpha}z = (a - ib)(x + iy) = ax + by + i(ay - bx),$$

so the stream function is

$$\Psi(z) = \operatorname{Im}(\overline{\alpha}z) = ay - bx,$$

and the streamlines are of the form

$$\operatorname{Im}(\overline{\alpha}z) = ay - bx = \text{constant}.$$

If $a \neq 0$, then these streamlines can be written in the form

$$y = \frac{b}{a}x + k, \quad \text{where } k \text{ is a constant},$$

which is a family of straight lines, each with gradient b/a, as in Figure 2.3.

The case $a = 0$ corresponds to the family of vertical lines given by

$$x = k, \quad \text{where } k \text{ is a constant}.$$

But how do we determine in which direction the fluid flows along these streamlines? The stream function does not tell us, and instead we need to use the velocity function q. Since $q(z) = \alpha = a + ib$, the flow direction along the streamlines is given by $\alpha = a + ib$. In Figure 2.3 the arrows indicate that a and b are both positive.

Our next example is slightly more complicated. First recall that any velocity function for an ideal flow must be the conjugate of an analytic function.

Flow near a stagnation point

The ideal flow with velocity function

$$q(z) = \overline{z} \quad (z \in \mathbb{C})$$

is an example of an ideal flow near a stagnation point.

In this flow the point 0 is distinguished by the fact that the velocity is zero there; that is, 0 is a stagnation point of the flow. As $q(z) = \overline{z}$ is not zero elsewhere, the streamline behaviour is more interesting elsewhere.

The conjugate velocity function is $\overline{q}(z) = z$, and a primitive of this is $\Omega(z) = \frac{1}{2}z^2$. Writing $z = x + iy$, we obtain

$$\tfrac{1}{2}z^2 = \tfrac{1}{2}(x + iy)^2 = \tfrac{1}{2}(x^2 - y^2) + ixy, \quad \text{so} \quad \operatorname{Im}\Omega(z) = xy.$$

Hence, by Theorem 2.1, the streamlines have equations of the form

$$xy = k, \quad \text{where } k \text{ is a constant}.$$

For $k \neq 0$ these are the hyperbolas shown in Figure 2.4; for $k = 0$ the streamlines form the real and imaginary axes.

Notice that, for each $k \neq 0$, the equation $xy = k$ determines *two* streamlines. For example, the equation $xy = 1$ determines one streamline in the upper-right quadrant and another in the lower-left quadrant.

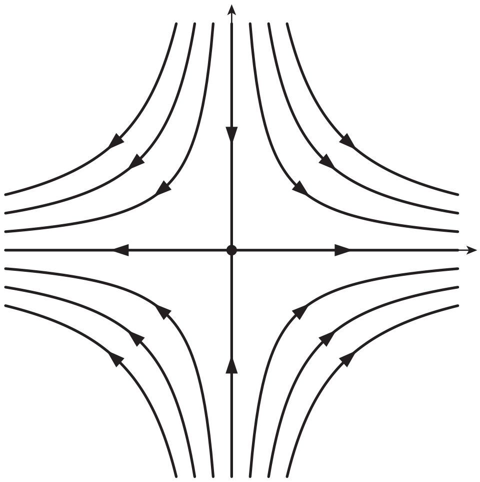


Figure 2.4 Streamlines near a stagnation point

Once again we use the formula for the velocity function, which is $q(z) = \overline{z} = x - iy$ in this case, to establish the direction of flow along the streamlines. For example, in the upper-right quadrant, where x and y are both positive, $q(z)$ points to the right and downwards. You can use a similar approach to determine the arrow directions in the other three quadrants, and also on each of the four half-axes.

On the axes there are five separate streamlines: on the positive real axis and the negative real axis fluid moves away from 0, whereas on the positive imaginary axis and the negative imaginary axis fluid moves towards 0 (but never reaches it since the velocity gets smaller and smaller as it approaches 0). The stagnation point 0 is a degenerate streamline since fluid at that point does not move.

Exercise 2.2

Determine a complex potential function for the velocity function

$$q(z) = -i\,\overline{z} \quad (z \in \mathbb{C}),$$

and use this to show that the streamlines for this flow have equations of the form

$$x^2 - y^2 = k, \quad \text{where } k \text{ is a constant.}$$

Our next examples are velocity functions defined on $\mathbb{C} - \{0\}$.

Flows with a source or sink

The ideal flow with velocity function

$$q(z) = \frac{1}{\overline{z}} = \frac{z}{|z|^2} \quad (z \in \mathbb{C} - \{0\})$$

is an example of a flow with a source.

The ideal flow with velocity function

$$q(z) = -\frac{1}{\overline{z}} = -\frac{z}{|z|^2} \quad (z \in \mathbb{C} - \{0\})$$

is an example of a flow with a sink.

First we discuss $q(z) = 1/\overline{z}$. You can see from this formula that the conjugate of $q(z)$ is an analytic function. The formula $q(z) = z/|z|^2$ for this velocity function shows that the direction of $q(z)$ is the same as the direction of z, which points radially outwards from 0. This suggests that the streamlines must be rays directed outwards from 0, and we can confirm this by finding a complex potential for the flow.

The conjugate velocity function in this example is $\overline{q}(z) = 1/z$. Here we must be careful as this function is analytic on its domain $\mathbb{C} - \{0\}$, but it does not have a primitive on the whole of this domain. However, we can take the principal logarithm function $\operatorname{Log} z$ as a primitive of $\overline{q}$ defined on $\mathbb{C}_\pi = \mathbb{C} - \{x : x \leq 0\}$, which is the complex plane cut along the negative real axis. Thus

$$\Omega(z) = \operatorname{Log} z = \log|z| + i \operatorname{Arg} z \quad (z \in \mathbb{C}_\pi).$$

Therefore the stream function is

$$\Psi(z) = \operatorname{Im} \Omega(z) = \operatorname{Arg} z, \quad \text{for } z \in \mathbb{C}_\pi,$$

so, by Theorem 2.1, the streamlines within $\mathbb{C}_\pi$ have equations of the form

$$\operatorname{Arg} z = \text{constant}.$$

These are indeed rays emerging from 0, as shown in Figure 2.5.

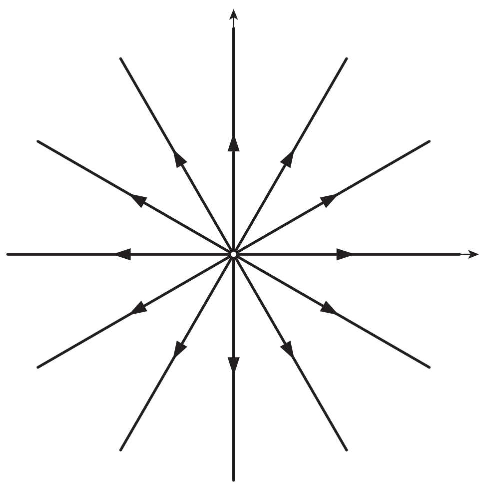


Figure 2.5 Radial streamlines from a source

Actually, this complex potential does not give us the streamline $\operatorname{Arg} z = \pi$, the negative real axis, because this ray is not in the domain of Log. However, this streamline would have appeared if we had used a generalised logarithm function such as $\operatorname{Log}_{2\pi}$ whose domain included that ray.

Since $q(z)$ points outwards from 0, we have outwards pointing arrows on the streamlines.

This example in which all streamlines move radially outwards is an example of a flow with a source, defined in Subsection 1.3, which behaves as if fluid is being fed into the system from a source at 0. Recall that the rate at which this happens is the source strength. This is the flux across a simple-closed contour Γ surrounding 0, given by Theorem 1.2 as

$$\operatorname{Im} \int_\Gamma \overline{q}(z)\, dz.$$

Since $\overline{q}(z) = 1/z$, Cauchy's Integral Formula gives the value $2\pi i$ for this integral, whatever simple-closed contour surrounding 0 is taken, so in this case 0 is a source of strength 2π.

Using the velocity function $q(z) = -1/\overline{z}$ reverses the flow that we have just seen. The streamlines are identical but the direction of flow is inwards rather than outwards. Now the point 0 is a sink for the flow, and the flux across the simple-closed contour Γ is -2π. Hence 0 is a sink of strength 2π.

More generally, if the velocity function is $q(z) = c/\overline{z}$, where $c \in \mathbb{R} - \{0\}$, then a complex potential function for the flow on $\mathbb{C}_\pi$ is $c \operatorname{Log} z$, and 0 is a source of strength $2\pi c$ if $c > 0$, and a sink of strength $-2\pi c$ if $c < 0$.

Our next example is also a velocity function defined on $\mathbb{C} - \{0\}$.

Flow around a vortex

The ideal flow with velocity function

$$q(z) = \frac{i}{\overline{z}} = \frac{iz}{|z|^2} \quad (z \in \mathbb{C} - \{0\})$$

is an example of a flow around a vortex.

For this velocity function the direction of flow is the same as the direction of iz, which is the same as the direction of z rotated anticlockwise through an angle of $\pi/2$. So $q(z)$ is directed perpendicular to the ray from 0 through z, and it seems likely that the streamlines are circles centred at 0 with the fluid moving around them anticlockwise.

The conjugate velocity function in this example is $\overline{q}(z) = -i/z$. Using the same primitive of $1/z$ as before, namely $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$, we find that

$$\Omega(z) = -i \operatorname{Log} z = \operatorname{Arg} z - i \log|z| \quad (z \in \mathbb{C}_\pi),$$

so

$$\operatorname{Im} \Omega(z) = -\log|z|, \quad \text{for } z \in \mathbb{C}_\pi,$$

and this function is constant whenever $|z|$ is constant. This confirms that the streamlines are circles centred at 0, as in Figure 2.6.

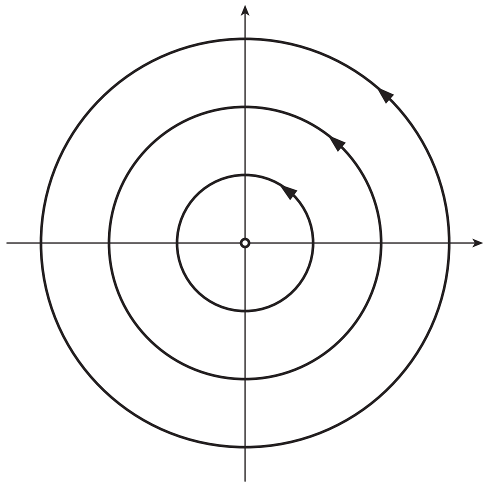


Figure 2.6 Streamlines around a vortex

This is an example of a flow with a vortex. In this case the flux across any simple-closed contour around 0 is zero. However, there is some circulation, which we use to measure the vortex strength. This is the (absolute value of) the circulation along a simple-closed contour Γ surrounding 0, given by Theorem 1.2 as

$$\operatorname{Re} \int_\Gamma \overline{q}(z)\, dz,$$

and in this case we again obtain the value 2π. Since $2\pi > 0$, the vortex 0 is an anticlockwise vortex. We can obtain a clockwise vortex flow by starting with $q(z) = -i/\overline{z}$.

A similar calculation shows that if the velocity function is $q(z) = ic/\overline{z}$, where $c \in \mathbb{R} - \{0\}$, then a complex potential for the flow on $\mathbb{C}_\pi$ is $-ic \operatorname{Log} z$, and 0 is a vortex of strength $|2\pi c|$.

The next exercise shows how, by taking a linear combination of the simple velocity functions studied so far, you can describe another type of fluid flow.

Exercise 2.3

Determine a complex potential function for the velocity function

$$q(z) = \frac{-1+8i}{\overline{z}} \quad (z \in \mathbb{C} - \{0\}).$$

Hence sketch the streamlines for this flow and determine the corresponding sink strength and vortex strength.

(*Hint*: To get a full picture of the streamlines, you will need to consider more than one complex potential function.)

2.3 A doublet in a uniform stream

Our next example of a velocity function is again defined on $\mathbb{C} - \{0\}$, but it gives a new type of fluid behaviour. You will see from the streamlines that this flow appears to be the result of combining features of both a source and a sink.

Flow due to a doublet

The ideal flow with velocity function

$$q(z) = -\frac{1}{\overline{z}^2} = -\frac{z^2}{|z|^4} \quad (z \in \mathbb{C} - \{0\})$$

is an example of a flow due to a **doublet**, or **dipole**.

It is not immediately obvious what the streamlines are from the velocity function, so we will obtain a complex potential. Since $\overline{q}(z) = -1/z^2$, we can take

$$\Omega(z) = \frac{1}{z} = \frac{\overline{z}}{|z|^2} \quad (z \in \mathbb{C} - \{0\}),$$

and hence, with $z = x + iy$, we have

$$\Psi(z) = \operatorname{Im}\Omega(z) = -\frac{y}{x^2 + y^2}, \quad \text{for } z \in \mathbb{C} - \{0\}.$$

This equation shows that, by Theorem 2.1, the streamlines have equations of the form

$$x^2 + y^2 = ky \quad \text{or} \quad y = 0,$$

where k is a constant. The equation $y = 0$ determines two streamlines: the positive real axis and the negative real axis. The equation $x^2 + y^2 = ky$ can be written in the form

$$x^2 + \left(y - \tfrac{1}{2}k\right)^2 = \tfrac{1}{4}k^2,$$

which is the equation of a circle with centre at $\frac{1}{2}ki$ and radius $\frac{1}{2}|k|$. This family of streamlines is shown in Figure 2.7.

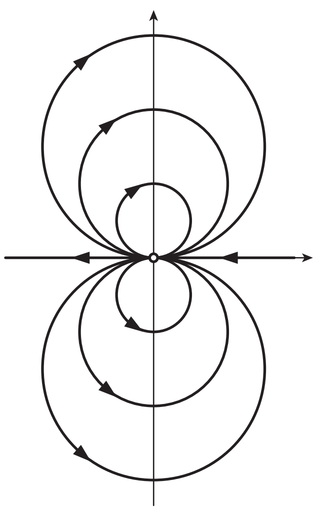


Figure 2.7 Streamlines near a doublet

The directions of the arrows can be found by using the formula for $q(z)$ to check the directions at simple points such as those on the axes (for example, $q(1) = -1$ and $q(i) = 1$), and by using the continuity of q.

The behaviour of a flow near a doublet combines that of a source and a sink, since fluid appears to be fed into the system from the origin to the left half-plane and simultaneously sucked out of the system from the right half-plane.

Our next example is the combination of a uniform flow (also called a uniform stream) and a doublet.

Flow due to a doublet in a uniform stream

The ideal flow with velocity function

$$q(z) = 1 - \frac{a^2}{\overline{z}^2} \quad (z \in \mathbb{C} - \{0\}),$$

where a is a positive constant, is an example of a flow due to a **doublet in a uniform stream**.

This is an important example, as you will see in Section 4, and in this case it is quite difficult to find the streamlines explicitly.

Far from 0, the term $-a^2/\overline{z}^2$ in the velocity function is insignificant compared to the term 1, so here the flow should closely resemble a uniform flow with velocity 1. Near to 0, however, it is the term $-a^2/\overline{z}^2$ that dominates, so the flow behaviour close to 0 should resemble that of a doublet. That leaves the interesting question of how these two flow regimes join up.

We can immediately find any stagnation points of the flow, that is, points where q vanishes. Observe that

$$q(z) = 0 \quad \iff \quad \overline{z} = \pm a.$$

So there is a pair of stagnation points on the real axis, one on either side of 0, at $\pm a$. These stagnation points make sense physically as they are points where the velocity of the uniform flow (directed to the right) is exactly balanced by the flow (directed to the left) due to the doublet.

Also, a property of both the uniform flow and the doublet is their symmetry under reflection in the real axis, so we would expect this property to be true in this example.

To look at the streamlines more closely, we find a complex potential. The conjugate velocity function is

$$\overline{q}(z) = 1 - \frac{a^2}{z^2} \quad (z \in \mathbb{C} - \{0\}),$$

so we can take

$$\Omega(z) = z + \frac{a^2}{z} \quad (z \in \mathbb{C} - \{0\}).$$

Hence, with $z = x + iy$, we have

$$\Psi(z) = \operatorname{Im}\Omega(z) = y\left(1 - \frac{a^2}{x^2 + y^2}\right).$$

The streamline equation is obtained by setting this expression equal to a constant k. If $k = 0$, then there are two possible solutions:

$$y = 0 \quad \text{or} \quad x^2 + y^2 = a^2.$$

The corresponding streamlines form four intervals of the real axis, the upper and lower halves of the circle with centre at 0 and radius a, and the two stagnation points $\pm a$. If $k \neq 0$, then we can rewrite the stream equation in the form

$$x = \pm\sqrt{\frac{a^2}{1 - k/y} - y^2}.$$

It is not easy to plot the streamlines using this equation, but we now have a good qualitative understanding of the overall streamline behaviour, as shown in Figure 2.8.

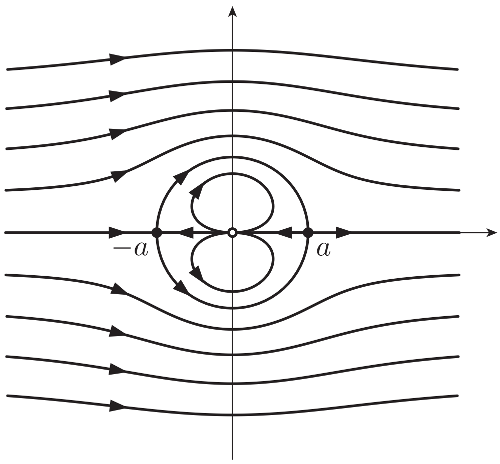


Figure 2.8 A doublet in a uniform stream

Roughly speaking, outside the circle $|z| = a$ this flow resembles a uniform flow, and inside that circle it resembles a doublet flow.

Finally, we point out that in any fluid flow there is no flux across a streamline, so we can treat any streamline as the solid boundary of a flow lying on either side of that streamline. For example, we can treat the circle $|z| = a$ as a solid circular boundary in Figure 2.8, so the fluid flow outside that circle can be interpreted as the result of placing a solid cylinder with circular cross-section in a uniform stream in three dimensions. This idea will be taken up and developed in Section 4 where we study the fluid flow past an obstacle.

The next two exercises show that the doublet flow can be thought of as the limit of a process in which a separate source and sink are brought ever closer together. Part (b) of Exercise 2.4 is challenging and depends on some facts about triangles and circles from Euclidean geometry.

Exercise 2.4

(a) Let $h > 0$. Show that the velocity function

$$q(z) = \frac{1}{\overline{z}} - \frac{1}{\overline{z} - h} \quad (z \in \mathbb{C} - \{0, h\})$$

has a complex potential function

$$\Omega(z) = \operatorname{Log} z - \operatorname{Log}(z - h) \quad (z \in \mathbb{C} - \{x \in \mathbb{R} : x \leq h\})$$

and stream function

$$\Psi(z) = \operatorname{Arg} z - \operatorname{Arg}(z - h) \quad (z \in \mathbb{C} - \{x \in \mathbb{R} : x \leq h\}).$$

(This represents the flow resulting from the combination of a source at the origin and a sink at $z = h$, both of strength 2π.)

(b) Show that the streamlines for this flow are circular arcs terminating at $z = 0$ and $z = h$, with the centres of the circles lying on the vertical line $x = \frac{1}{2}h$, and also three intervals of the real axis. Hence draw a rough sketch of the streamlines for this flow.

(*Hint*: Use the fact from elementary geometry that if a line segment subtends equal angles at two or more points, then these points and the two endpoints of the line segment all lie on a common circle.)

Exercise 2.5

The velocity function

$$q_h(z) = \frac{1}{h}\left(\frac{1}{\overline{z}} - \frac{1}{\overline{z} - h}\right) \quad (z \in \mathbb{C} - \{0, h\}),$$

where $h > 0$, represents the same type of flow as in Exercise 2.4, but with the strengths of the source and sink now equal to $2\pi/h$, which is inversely proportional to the distance between them.

Show that

$$\lim_{h \to 0} q_h(z)$$

is the velocity function for a doublet.

Further exercises

Exercise 2.6

Consider the ideal flow with velocity function

$$q(z) = 2 + \frac{2}{\overline{z}} \quad (z \in \mathbb{C} - \{0\}),$$

which is a uniform flow on which is superposed a source at 0.

(a) Write down a complex potential function and a stream function for this flow.

(b) Calculate the only stagnation point for this flow.

(c) Given that there is a streamline for this flow that approaches this stagnation point from the upper half-plane, find the equation of this streamline, and find the equation of a second streamline that approaches the stagnation point from the lower half-plane.

(d) Determine how the two streamlines found in part (c) behave for large values of z. Use this information to draw a rough sketch of the streamlines for this flow.

Exercise 2.7

Prove that if the velocity function for an ideal flow with flow region $\mathbb{C}$ is bounded, then the flow is uniform.

(*Hint*: Use Liouville's Theorem (Theorem 2.2 of Unit B2).)

3 The Joukowski functions

After working through this section, you should be able to:

- define the basic *Joukowski function* J and the family of Joukowski functions J_α, where α is a non-zero complex number
- understand some of the properties of the Joukowski functions
- describe some of the properties of the inverse functions J^{-1} and J_α^{-1}
- appreciate why the images of certain circles under a Joukowski function have an aerofoil shape, and how this aerofoil shape is related to the *critical points* of the function.

In this section we study in detail a family of analytic functions that are of particular relevance to the analysis of fluid flow past a wing-shaped object. We then apply these functions to solve various fluid flow problems in the final sections of the unit.

3.1 The basic Joukowski function

We start with the basic **Joukowski function** (where Joukowski is pronounced 'jew-coff-ski')

$$J(z) = z + \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}).$$

On its domain $\mathbb{C} - \{0\}$ this function is not a conformal mapping, since its derivative is 0 at the two points ± 1. However, by restricting the domain of J to the region $\{z : |z| > 1\}$ we do obtain a useful conformal mapping, as shown in the following theorem. The proof of part (c) is challenging and may be omitted on a first reading.

Theorem 3.1

The function $J(z) = z + 1/z$ has the following properties.

(a) J maps the circle $\{z : |z| = 1\}$ onto the line segment $[-2, 2]$, with $J(1) = 2$ and $J(-1) = -2$.

(b) J maps the region $\{z : |z| > 1\}$ conformally onto the region $\mathbb{C} - [-2, 2]$.

(c) The restriction of J to $\{z : |z| > 1\}$ has inverse function

$$J^{-1}(w) = \tfrac{1}{2}(w + w\sqrt{1 - 4/w^2}) \quad (w \in \mathbb{C} - [-2, 2]).$$

(d) J has a non-vanishing derivative at all points of $\mathbb{C} - \{0\}$ except $z = \pm 1$.

Proof We prove properties (a) and (b) at the same time by expressing the function J as a composition of simple conformal mappings, following the approach used in Subsection 4.3 of Unit C3. To do this, we first rewrite the equation

$$w = z + \frac{1}{z}$$

in an equivalent form. We have

$$w + 2 = z + 2 + \frac{1}{z} = \frac{z^2 + 2z + 1}{z} = \frac{(z+1)^2}{z}$$

and

$$w - 2 = z - 2 + \frac{1}{z} = \frac{z^2 - 2z + 1}{z} = \frac{(z-1)^2}{z},$$

so

$$\frac{w+2}{w-2} = \left(\frac{z+1}{z-1}\right)^2.$$

Rearranging this equation, we obtain

$$w = \frac{2\left(\dfrac{z+1}{z-1}\right)^2 + 2}{\left(\dfrac{z+1}{z-1}\right)^2 - 1}.$$

This equation shows that J can be expressed as a composition of three simpler functions:

$$z_1 = \frac{z+1}{z-1}, \quad z_2 = z_1^2, \quad w = \frac{2z_2+2}{z_2-1}.$$

The composition of these three functions is illustrated in Figure 3.1.

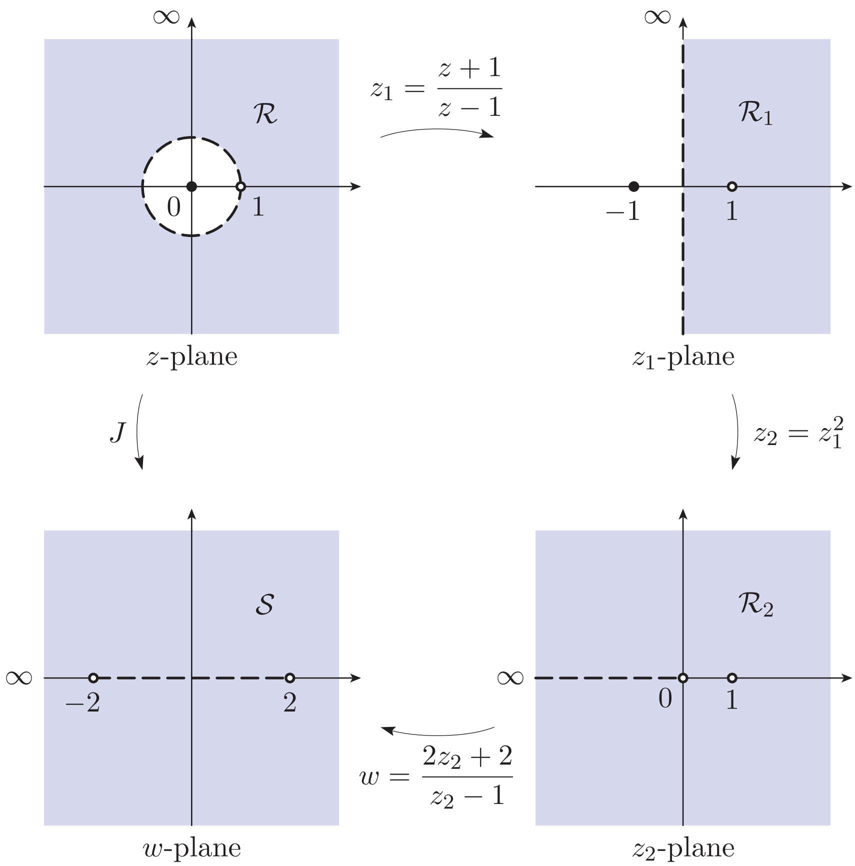


Figure 3.1 The Joukowski function as a composition of simpler functions

The region $\mathcal{R} = \{z : |z| > 1\}$ in the top left-hand corner of Figure 3.1 has a boundary in $\widehat{\mathbb{C}}$ that consists of the unit circle together with the point at infinity. The points 0 and ∞ are inverse points with respect to the unit circle. The Möbius transformation

$$z_1 = \frac{z+1}{z-1}$$

sends 0 and ∞ to the points -1 and 1, respectively, which are inverse points with respect to the image boundary. Also, the point 1 on the unit circle is sent to ∞, so the unit circle must map onto the extended imaginary axis. The boundary point of $\mathcal{R}$ at ∞ is mapped to 1, and -1 is mapped to 0. Hence, by Theorem 4.1 of Unit C3, the region $\mathcal{R}$ is mapped onto the right half-plane with the point 1 removed, which is the region $\mathcal{R}_1 = \{z_1 : \operatorname{Re} z_1 > 0\} - \{1\}$.

Next we apply the square function $z_2 = z_1^2$ to send $\mathcal{R}_1$ onto the standard cut plane $\mathbb{C}_\pi = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ with the point 1 removed, which is the region $\mathcal{R}_2 = \mathbb{C}_\pi - \{1\}$.

Finally, the Möbius transformation

$$w = \frac{2z_2 + 2}{z_2 - 1}$$

maps the 'missing' point at 1 to ∞, and it maps the points 0 and ∞ to the points -2 and 2, respectively. Thus the extended real line is mapped onto itself, and the negative real axis together with 0 and ∞ is mapped onto the interval $[-2, 2]$. Since any Möbius transformation is a one-to-one mapping from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$, we see that the image of the region $\mathcal{R}_2$ under this Möbius transformation is the region $\mathcal{S} = \mathbb{C} - [-2, 2]$, as required.

Overall, therefore, J maps the unit circle onto the interval $[-2, 2]$, and since all the mappings in Figure 3.1 are one-to-one and conformal on the given regions, J is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

To prove part (c) we need to find a formula for the inverse function of this restriction of the Joukowski function. Suppose then that $w = J(z)$, where $w \in \mathcal{S}$ and $z \in \mathcal{R}$. If we try to solve the equation

$$w = J(z) = z + 1/z, \quad \text{that is, } z^2 - wz + 1 = 0,$$

to obtain z in terms of w by using the quadratic formula, then we obtain

$$z = \tfrac{1}{2}\left(w \pm \sqrt{w^2 - 4}\right),$$

and unfortunately it is not clear that this formula gives an analytic function, no matter which square root we use. Instead we solve the equation $z^2 - wz + 1 = 0$ in a different way.

After completing the square, this quadratic equation is equivalent to

$$\left(z - \tfrac{1}{2}w\right)^2 = \tfrac{1}{4}w^2 - 1.$$

Then on multiplying both sides of this equation by $(2/w)^2 = 4/w^2$, we obtain

$$(2z/w - 1)^2 = 1 - 4/w^2.$$

Now, on the region $\mathcal{S}$, the function $w \longmapsto w^2$ maps $\mathcal{S}$ onto $\mathbb{C} - [0, 4]$, so the function $w \longmapsto 1 - 4/w^2$ maps $\mathcal{S}$ onto $\mathbb{C}_\pi$. Since the principal square root function is analytic on $\mathbb{C}_\pi$, the function $w \longmapsto \sqrt{1 - 4/w^2}$ is analytic on the region $\mathcal{S}$. Thus, for $w \in \mathcal{S}$ and $z \in \mathcal{R}$ with $w = J(z)$, we have

$$2z/w - 1 = \pm\sqrt{1 - 4/w^2}, \quad \text{that is, } z = \tfrac{1}{2}\left(w \pm w\sqrt{1 - 4/w^2}\right),$$

and either choice of sign in this formula gives the rule for an analytic function on $\mathcal{S}$.

Since the inverse function we seek is a one-to-one mapping of the interval $(2, \infty)$ onto the interval $(1, \infty)$, we deduce that the choice of sign in the above formula must be $+$, at least for values of w on the interval $(2, \infty)$. Therefore, by the Uniqueness Theorem (Theorem 5.5 of Unit B3), the inverse function must have the rule

$$w \longmapsto \tfrac{1}{2}\left(w + w\sqrt{1 - 4/w^2}\right), \quad \text{for } w \in \mathcal{S}.$$

Part (d) holds because

$$J'(z) = 1 - \frac{1}{z^2}, \quad \text{so } J'(z) = 0 \iff z = \pm 1.$$

We now discuss how the function J behaves geometrically on the region $\mathcal{R} = \{z : |z| > 1\}$.

For values of z such that $|z|$ is large, the quantity $1/z$ is close to 0, so $J(z) = z + 1/z$ is close to z. This implies that such points z are not moved far by the function J.

However, the behaviour of J near the unit circle is more interesting. It may seem strange that the function J maps the unit circle onto a line segment, a set that is 'pointed' at both ends. This property can be linked to the fact that the endpoints of the line segment, 2 and -2, are the images of the points 1 and -1, respectively, and the derivative of J is zero at 1 and -1.

Recall that if f is an analytic function and α is a point of its domain at which $f'(\alpha) \neq 0$, then f is conformal at α; that is, f preserves the angle between any two smooth paths that meet at α (Theorem 4.2 of Unit A4). However, if α is a point for which $f'(\alpha) = 0$, a so-called **critical point** of f, and $f''(\alpha) \neq 0$, then the effect of f is to *double* the angle between two smooth paths emerging from α; see the remark after the proof of the Local Mapping Theorem (Theorem 3.2 of Unit C2).

This helps to explain why the image of the unit circle under J is pointed at both ends. The unit circle can be thought of as a pair of semicircular smooth paths meeting at the points $z = \pm 1$, with an angle π between them at each of these points. So we can think of the images of these two paths as two copies of the line segment between 2 and -2 meeting at an angle 2π at each end, producing the pointed image.

Another way to understand the effect of the function J on the unit circle is to parametrise this circle as

$$z = e^{it}, \quad 0 \leq t \leq 2\pi.$$

Then

$$\begin{aligned} J(z) = J(e^{it}) &= e^{it} + 1/e^{it} \\ &= (\cos t + i\sin t) + (\cos t - i\sin t) \\ &= 2\cos t. \end{aligned}$$

So, as t increases from 0 to 2π, the point $z = e^{it}$ traverses the unit circle once anticlockwise, starting and finishing at 1, and the image $J(z) = J(e^{it})$ traverses the line segment $[-2, 2]$ twice: first it moves from 2 to -2 (as t increases from 0 to π) and then it moves back from -2 to 2 (as t increases from π to 2π).

The function J maps points near the upper half of the unit circle to points just above the line segment $[-2, 2]$, whereas points near the lower half of the unit circle are mapped to points just below this line segment. The next exercise asks you to confirm this behaviour of J by calculating the effect of J on certain line segments *outside* the unit circle.

Exercise 3.1

(a) Show that if $r > 1$, then J is a one-to-one mapping of the real interval $[1, r]$ onto the real interval $[2, r + 1/r]$.

(b) Show that if $r > 1$, then J is a one-to-one mapping of the line segment on the imaginary axis with endpoints i and ir onto the line segment on the imaginary axis with endpoints 0 and $i(r - 1/r)$.

3.2 A family of Joukowski functions

In this subsection we introduce a general family of Joukowski functions in order to increase our range of possible applications to fluid flow. First we write

$$J_a(z) = z + \frac{a^2}{z} \quad (z \in \mathbb{C} - \{0\}),$$

where $a > 0$. The functions J_a are closely related to the basic Joukowski function $J = J_1$, and we can deduce the properties of these functions J_a by observing that

$$J_a(z) = z + \frac{a^2}{z} = a\left(z/a + \frac{1}{z/a}\right).$$

This equation tells us that $w = J_a(z)$ can be obtained by composing three functions:

- the scaling $z_1 = z/a$
- the basic Joukowski function $z_2 = J(z_1) = z_1 + 1/z_1$
- the scaling $w = az_2$.

This representation of J_a as the composition of two scalings with the function J allows us to deduce the following properties of J_a from the properties of J obtained in Theorem 3.1; we omit the proof.

Theorem 3.2

For $a > 0$, the function J_a has the following properties.

(a) J_a maps the circle $\{z : |z| = a\}$ onto the line segment $[-2a, 2a]$, with $J(a) = 2a$ and $J(-a) = -2a$.

(b) J_a maps the region $\{z : |z| > a\}$ conformally onto the region $\mathbb{C} - [-2a, 2a]$.

(c) The restriction of J_a to $\{z : |z| > a\}$ has inverse function

$$J_a^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4a^2/w^2}\right) \quad (w \in \mathbb{C} - [-2a, 2a]).$$

(d) J_a has a non-vanishing derivative at all points of $\mathbb{C} - \{0\}$ except $z = \pm a$.

The next exercise investigates the effect of the Joukowski function J_a on certain curves that lie outside the circle $\{z : |z| = a\}$. It is convenient to use the notation $C_r = \{z : |z| = r\}$, where $r > 0$, throughout the rest of this unit.

Exercise 3.2

(a) Show that if $r > a$, then J_a is a one-to-one mapping of the circle C_r onto the ellipse in the w-plane (where $w = u + iv$) with equation

$$\frac{u^2}{(r + a^2/r)^2} + \frac{v^2}{(r - a^2/r)^2} = 1,$$

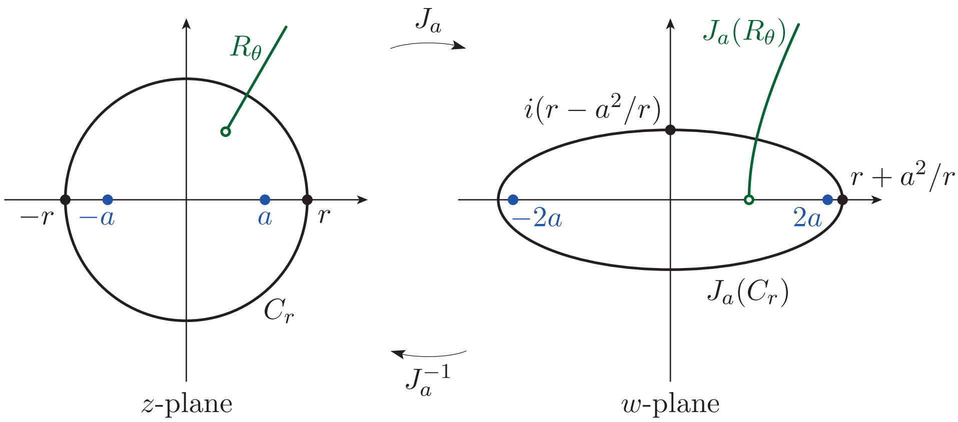
shown in the figure below.

(*Hint*: Use the parametrisation $z = re^{it}$, for $0 \leq t \leq 2\pi$, of C_r.)

(b) Suppose that $\theta \in (0, \pi/2)$. Show that J_a is a one-to-one mapping of the ray $R_\theta = \{te^{i\theta} : a < t < \infty\}$ onto the part of the hyperbola with equation

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4a^2$$

that lies in the upper-right quadrant.



Next we combine two results about the mapping properties of the Joukowski functions J_a in order to obtain insight into why these functions have relevance to studying the flow past a wing-shaped object.

The function $J_a(z) = z + a^2/z$ maps

- the circle $C_a = \{z : |z| = a\}$ onto the line segment $[-2a, 2a]$, as we saw in Theorem 3.2(a)
- the circle $C_r = \{z : |z| = r\}$, where $r > a$, onto the ellipse with equation

$$\frac{u^2}{(r + a^2/r)^2} + \frac{v^2}{(r - a^2/r)^2} = 1,$$

  as we saw in Exercise 3.2(a).

Consider now the circle

$$C = \{z : |z + b| = a + b\},$$

where $0 < a < r$ and $b = \frac{1}{2}(r - a) > 0$. This circle C, shown on the left-hand side of Figure 3.2, has centre $-b$ and radius $a + b$. It has the same tangent at $z = a$ as the circle C_a, so its image curve $J_a(C)$ can be expected to resemble that of $J_a(C_a) = [-2a, 2a]$ in the vicinity of the point $w = 2a$. In particular, $J_a(C)$ should be pointed at $w = 2a$; more precisely, it should have a *cusp* at that point, which means roughly speaking that the upper and lower parts of the image curve meet at an angle of 0. On the other hand, C has the same tangent at $z = -r$ as the circle C_r, so $J_a(C)$ can be expected to resemble the ellipse $J_a(C_r)$ in the vicinity of $w = -(r + a^2/r)$, as shown in Figure 3.2.

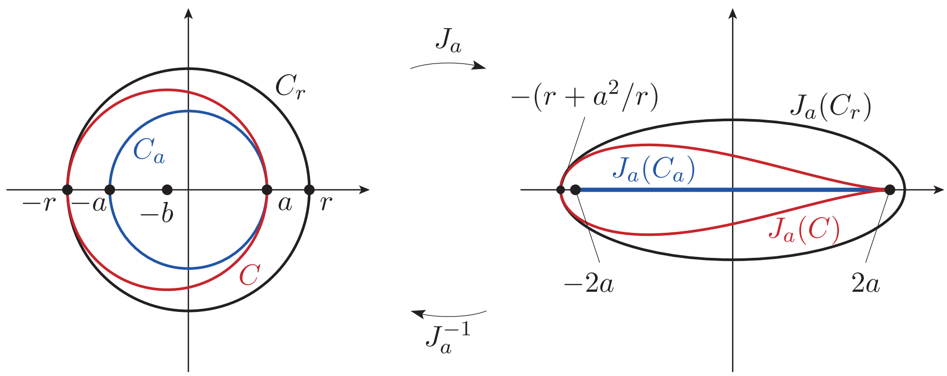


Figure 3.2 A Joukowski function mapping a circle onto an aerofoil

At all points of C other than $z = a$ we have $J_a'(z) \neq 0$, so the image contour is smooth except at the single image point $w = 2a$. Also, the mapping J_a preserves the symmetry of C under reflection in the real axis, since $J_a(\overline{z}) = \overline{J_a(z)}$, for all $z \in \mathbb{C} - \{0\}$.

These features of $J_a(C)$ can be seen on the right of Figure 3.2. The wing-shaped set with boundary $J_a(C)$ is an example of a type of set called a *Joukowski aerofoil*. Such sets are described in more detail in Section 5, where we model the fluid flow past them using the function J_a.

In order to model the fluid flow past aerofoils that are inclined at an angle to the real axis, we extend the family of Joukowski functions J_a by allowing the parameter to be any non-zero complex number; that is, we consider the functions

$$J_\alpha(z) = z + \frac{\alpha^2}{z},$$

where α is any non-zero complex number.

In the same way that we expressed $w = J_a(z)$ as the composition of three simpler functions, we can decompose the equation $w = J_\alpha(z)$ as follows:

- the complex scaling $z_1 = z/\alpha$
- the basic Joukowski function $z_2 = J(z_1) = z_1 + 1/z_1$
- the complex scaling $w = \alpha z_2$.

This composition allows us to deduce the following properties of J_α from Theorem 3.1; once again we omit the proof. Here and later in the unit we use the notation $L(\alpha,\beta)$ to denote the closed line segment in $\mathbb{C}$ with endpoints α and β.

Theorem 3.3

For $\alpha \in \mathbb{C} - \{0\}$, the function J_α has the following properties.

(a) J_α maps the circle $\{z : |z| = |\alpha|\}$ onto the line segment $L(-2\alpha, 2\alpha)$, with $J(\alpha) = 2\alpha$ and $J(-\alpha) = -2\alpha$.

(b) J_α maps the region $\{z : |z| > |\alpha|\}$ conformally onto the region $\mathbb{C} - L(-2\alpha, 2\alpha)$.

(c) The restriction of J_α to $\{z : |z| > |\alpha|\}$ has inverse function

$$J_\alpha^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4\alpha^2/w^2}\right) \quad (w \in \mathbb{C} - L(-2\alpha, 2\alpha)).$$

(d) J_α has a non-vanishing derivative at all points of $\mathbb{C} - \{0\}$ except $z = \pm\alpha$.

There is another way to write $w = J_\alpha(z)$ as the composition of three functions, which will prove useful in the next section. For this method we let $\phi = -\operatorname{Arg}\alpha$; that is, $\alpha = |\alpha|e^{-i\phi}$, where $-\pi \le \phi < \pi$. Then

$$J_\alpha(z) = z + \frac{\alpha^2}{z} = e^{-i\phi}\left(ze^{i\phi} + \frac{|\alpha|^2}{ze^{i\phi}}\right),$$

so $w = J_\alpha(z)$ is a composition of the three functions

- $z_1 = ze^{i\phi}$ (a rotation about 0 through the angle ϕ)
- $z_2 = J_{|\alpha|}(z_1) = z_1 + |\alpha|^2/z_1$ (a mapping J_a, with $a = |\alpha|$)
- $w = e^{-i\phi}z_2$ (a rotation about 0 through the angle $-\phi$).

Thus if we denote by R_ϕ the rotation about 0 through the angle ϕ, then

$$J_\alpha = R_\phi^{-1} \circ J_{|\alpha|} \circ R_\phi. \tag{3.1}$$

The next exercise asks you to use this decomposition of J_α to find the image of a circle under the function J_α for a particular complex number α.

Exercise 3.3

Use the result of Exercise 3.2(a) together with equation (3.1) to show that the image of the circle $C_r = \{z : |z| = r\}$, where $0 < a < r$, under the mapping J_{ia}, is the ellipse

$$E = \left\{u + iv : \frac{u^2}{(r - a^2/r)^2} + \frac{v^2}{(r + a^2/r)^2} = 1\right\},$$

illustrated in Figure 3.3, and that J_{ia} maps the region $\{z : |z| > r\}$ onto the unbounded region with E as its boundary.

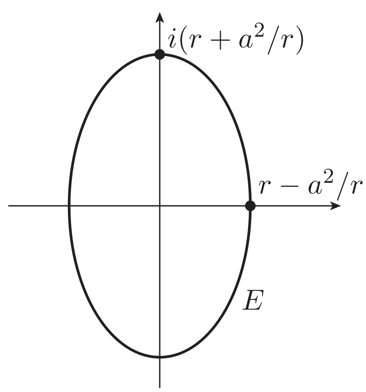


Figure 3.3 The ellipse E

Figure 3.4 illustrates the decomposition of J_α given in equation (3.1), and also illustrates parts (a) and (b) of Theorem 3.3.

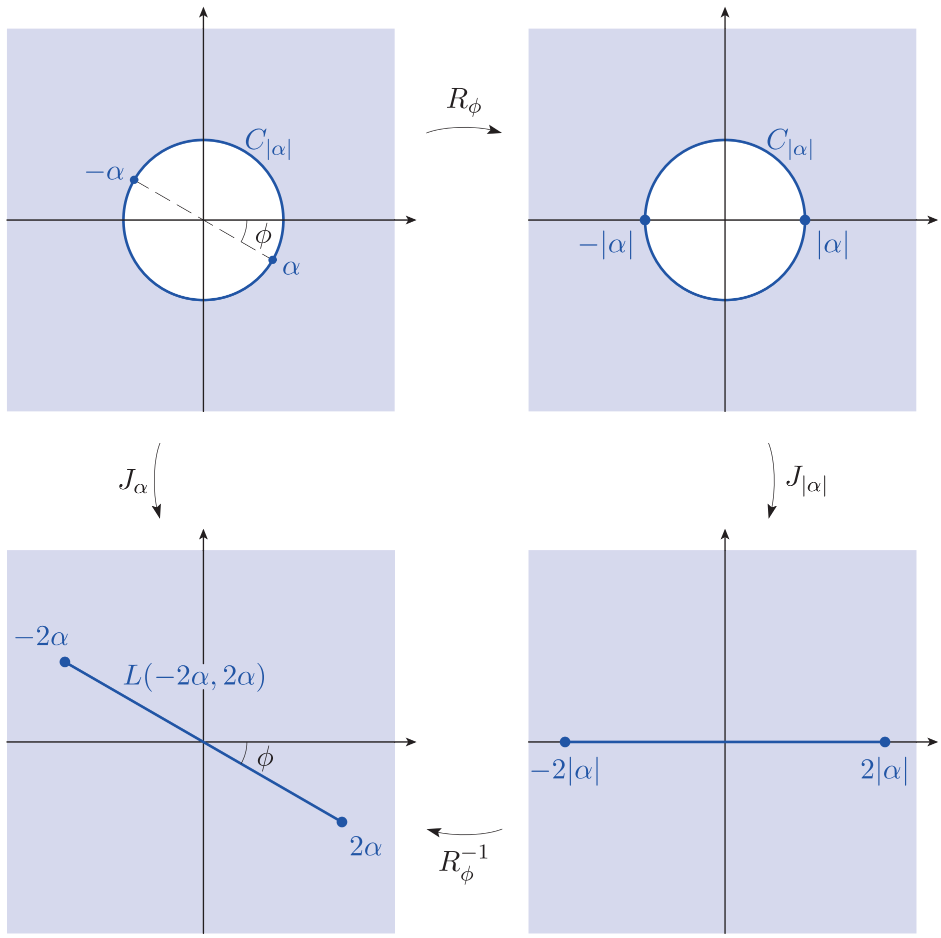


Figure 3.4 Decomposition of a Joukowski function J_α, where $\operatorname{Arg}\alpha < 0$

Further exercises

Exercise 3.4

Consider the set

$$K = [-2,-1] \cup \{z : |z| \leq 1\} \cup [1,2],$$

consisting of a closed disc of radius 1 with two line segments attached, illustrated in Figure 3.5.

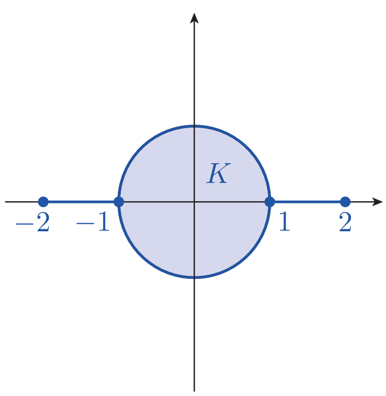


Figure 3.5 The set K

Use the result of Exercise 3.1(a) to show that the Joukowski function J is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - \left[-\frac{5}{2}, \frac{5}{2}\right]$.

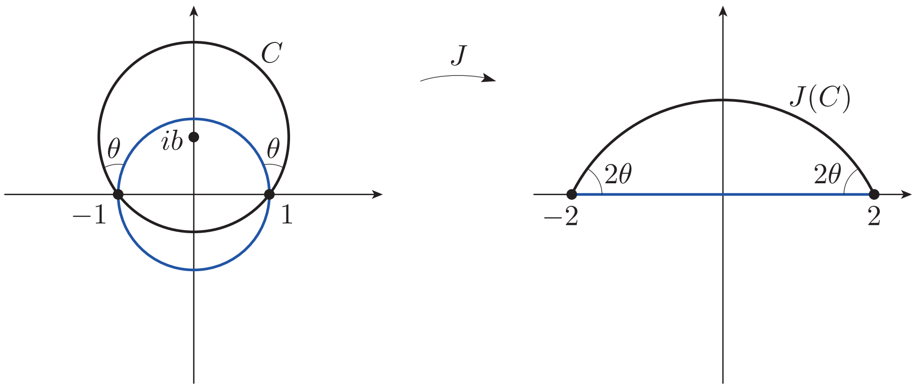
The final exercise in this section is challenging; its result will be needed for one of the further exercises of Section 5.

Exercise 3.5

Let C be the circle centred at ib, where $b > 0$, that passes through the two points ± 1, making an angle $\theta \in (0, \pi/2)$ with the unit circle at each point.

Prove that the image of the circle C under J is a circular arc in the upper half-plane with endpoints ± 2, as shown in the figure.

(*Hint*: Use the decomposition of the basic Joukowski function $J(z) = z + 1/z$ given in the proof of Theorem 3.1, and remember that Möbius transformations preserve angles whereas the function $z \longmapsto z^2$ doubles acute angles.)

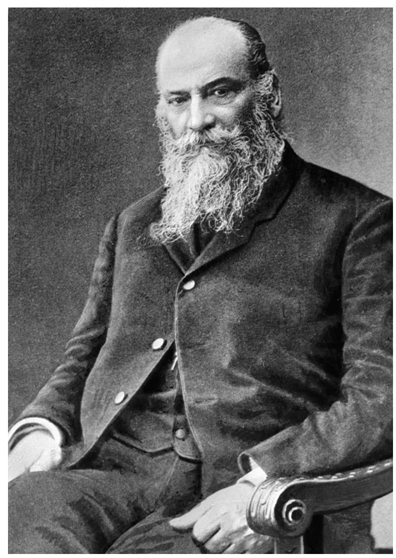


Origin of the Joukowski functions

In the late nineteenth century, the possibility of creating a powered flying machine was investigated by scientists, mathematicians and engineers in several countries. The Russian mathematician Nikolay Yegorovich Zhukovsky (1847–1921), whose surname is commonly written as Joukowski, worked extensively on fluid flows. In the 1880s he realised that complex analysis, and in particular the conformal mappings now named after him, could help us to understand how the flow of air past a suitably shaped object could cause 'lift', and so contribute to solving the problem of powered flight.

Using these mappings, Zhukovsky produced the first theoretical predictions of the lift due to the flow of air around an object with a rounded nose and a sharp trailing edge, and also tested his theory by building a wind tunnel. He is regarded as a key figure in Russian aviation history, and the city of Zhukovsky (near Moscow) and its international airport are named after him.

Historically, Zhukovsky's Russian name was written as Joukowski, or Joukowsky, when using the roman alphabet, and this is still the usual spelling for mathematical objects associated with him.



Nikolay Yegorovich Zhukovsky

4 Flow past an obstacle

After working through this section, you should be able to:

- explain the main features of the flow past a disc due to a uniform stream with circulation
- explain what is meant by an *obstacle*, and state the Obstacle Problem
- solve the Obstacle Problem for flows around certain symmetric obstacles, with or without circulation, by applying the Flow Mapping Theorem.

4.1 Flow past a circular cylinder

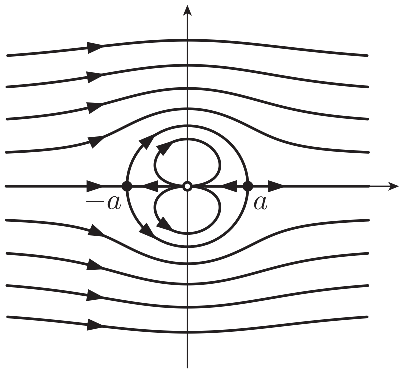


Figure 4.1 A doublet in a uniform stream

In this subsection we return to a flow considered in Subsection 2.3, a doublet in a uniform stream (see Figure 4.1), for which the velocity function is

$$q(z) = 1 - \frac{a^2}{\overline{z}^2} \quad (z \in \mathbb{C} - \{0\}), \tag{4.1}$$

and a corresponding complex potential function is

$$\Omega(z) = z + \frac{a^2}{z} \quad (z \in \mathbb{C} - \{0\}).$$

Here, and throughout this section, the letter a denotes a positive real number. As you will see, it is no accident that this formula for $\Omega(z)$ is closely related to the Joukowski function.

Exercise 4.1

Consider the flow with velocity function $q(z) = 1 - a^2/\overline{z}^2$ $(z \in \mathbb{C} - \{0\})$.

(a) Verify that the points $z = a$ and $z = -a$ are the only stagnation points of the flow.

(b) Show that

$$q(ae^{it}) = (-2i \sin t)e^{it}, \quad \text{for } 0 \le t \le 2\pi.$$

Hence derive the flow speed at the point $z = ae^{it}$ on the circle $\{z : |z| = a\}$.

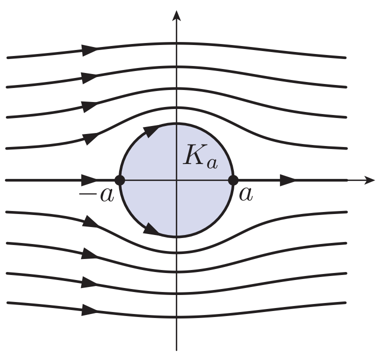


Figure 4.2 Uniform flow past the closed disc K_a

In Subsection 2.3 we claimed that equation (4.1) is the velocity function of a uniform flow past a solid cylinder with circular cross-section of radius a (see Figure 4.2); the cross-section of this cylinder can be taken to be the closed disc

$$K_a = \{z : |z| \le a\}, \quad \text{where } a > 0.$$

We use this notation for a closed disc with centre 0 and radius a throughout the rest of this unit.

This claim was based on the fact that the circle

$$\partial K_a = C_a = \{z : |z| = a\}$$

is made up of streamlines. However, there are other ideal flows for which the circle C_a is made up of streamlines. As you saw in Subsection 2.2, the complex potential function

$$\Omega(z) = -ic\,\mathrm{Log}\,z \quad (z \in \mathbb{C}_\pi),$$

where $c \in \mathbb{R} - \{0\}$, corresponds to the flow around a vortex with strength $|2\pi c|$, and all the streamlines are circles with centre 0. If $c > 0$ then the flow is anticlockwise, and if $c < 0$ then the flow is clockwise. Thus it is reasonable to guess that the complex potential function

$$\Omega_{a,c}(z) = z + \frac{a^2}{z} - ic\,\mathrm{Log}\,z \quad (z \in \mathbb{C}_\pi)$$

might correspond to the velocity function for an ideal flow for which C_a is made up of streamlines. This flow will play a major role in the rest of this unit, so we record its main properties in a theorem.

Theorem 4.1

For $a > 0$, $c \in \mathbb{R}$, the ideal flow with velocity function

$$q_{a,c}(z) = \overline{1 - \frac{a^2}{z^2} - \frac{ic}{z}} \quad (z \in \mathbb{C} - \{0\})$$

and complex potential function

$$\Omega_{a,c}(z) = z + \frac{a^2}{z} - ic\,\mathrm{Log}\,z \quad (z \in \mathbb{C}_\pi)$$

has the following properties:

(a) $\lim\limits_{z\to\infty} q_{a,c}(z) = 1$

(b) $\partial K_a = C_a$ is made up of streamlines for the flow

(c) for any simple-closed contour Γ surrounding K_a,

(i) $\mathcal{C}_\Gamma = \mathrm{Re}\displaystyle\int_\Gamma \overline{q_{a,c}}(z)\,dz = 2\pi c$

(ii) $\mathcal{F}_\Gamma = \mathrm{Im}\displaystyle\int_\Gamma \overline{q_{a,c}}(z)\,dz = 0.$

Remarks

1. Note that $\Omega_{a,c}(z)$ has domain the cut plane $\mathbb{C}_\pi$ (because its formula includes $\mathrm{Log}\,z$) and

$$\overline{\Omega'_{a,c}(z)} = q_{a,c}(z), \quad \text{for } z \in \mathbb{C}_\pi,$$

but $q_{a,c}$ has the larger domain $\mathbb{C} - \{0\}$.

2. We write $\lim\limits_{z\to\infty} q_{a,c}(z) = \beta$ to mean that

$$q_{a,c}(z) \to \beta \text{ as } z \to \infty.$$

Limits of this type were discussed in Subsection 1.3 of Unit C3.

3. Property (c)(ii) is in fact a consequence of property (b).

Proof of Theorem 4.1 Property (a) follows immediately from the formula for $q_{a,c}(z)$.

To verify property (b), first recall from Theorem 3.2 that J_a maps the circle C_a onto the interval $[-2a, 2a]$. Therefore, for $z \in C_a$ and $z \neq -a$, we have

$$\begin{aligned}
\operatorname{Im} \Omega_{a,c}(z) &= \operatorname{Im}(z + a^2/z - ic \operatorname{Log} z) \\
&= \operatorname{Im}(J_a(z) - ic \operatorname{Log} z) \\
&= 0 + \operatorname{Im}(-ic \operatorname{Log} z) \\
&= -\operatorname{Im}(ic(\log|z| + i \operatorname{Arg} z)) \\
&= -c \log a.
\end{aligned}$$

Since $-c \log a$ is a constant, the set C_a is made up of streamlines for $q_{a,c}$, by Theorem 2.1. (To include $z = -a$ on a streamline, we can consider a generalised logarithm function with a different cut plane, as discussed for a flow with a source in Subsection 2.2.)

To verify property (c), we use the Residue Theorem to obtain

$$\begin{aligned}
\int_\Gamma \overline{q_{a,c}}(z)\, dz &= \int_\Gamma \left(1 - \frac{a^2}{z^2} - \frac{ic}{z}\right) dz \\
&= 2\pi i(-ic) = 2\pi c.
\end{aligned}$$

Hence $\mathcal{C}_\Gamma = 2\pi c$ and $\mathcal{F}_\Gamma = 0$, by Theorem 1.2. ■

The vortex term $-ic \operatorname{Log} z$, which gives rise to the non-zero circulation $2\pi c$ around Γ, affects the positions of the stagnation points of the flow, as we now ask you to verify. For the case $c = 0$, these points were found in Exercise 4.1.

Exercise 4.2

Find the stagnation points of the flow with velocity function

$$q_{a,c}(z) = \overline{1 - \frac{a^2}{z^2} - \frac{ic}{z}} \quad (z \in \mathbb{C} - \{0\}),$$

where $a > 0$ and $c \in \mathbb{R}$. Describe the locations of these stagnation points when

(a) $-2a < c < 0$, (b) $c = -2a$, (c) $c < -2a$.

Theorem 4.1 allows us to think of $q_{a,c}$ as the velocity function obtained by placing the 'obstacle' K_a into a uniform stream (with $q(z) = 1$) and requiring that there is no flow of fluid across any part of ∂K_a, and that the circulation of the flow around K_a is $2\pi c$.

Each streamline for this flow satisfies an equation of the form

$$\begin{aligned}
\operatorname{Im} \Omega_{a,c}(z) &= \operatorname{Im}(z + a^2/z) + \operatorname{Im}(-ic \operatorname{Log} z) \\
&= y\left(1 - \frac{a^2}{x^2 + y^2}\right) - c \log(x^2 + y^2)^{1/2} \\
&= \text{constant}.
\end{aligned}$$

Unfortunately it is not possible to solve these equations (for y in terms of x, or vice versa) when $c \neq 0$. The streamline diagrams are shown in Figure 4.3 for various values of $c \leq 0$. The directions of the arrows can be obtained from the formula for $q_{a,c}(z)$; a quick method for doing this is to start with $q_{a,c}(z) \approx 1$ when $|z|$ is large, and then use the continuity of $q_{a,c}$ to determine the directions when z is nearer to K_a.

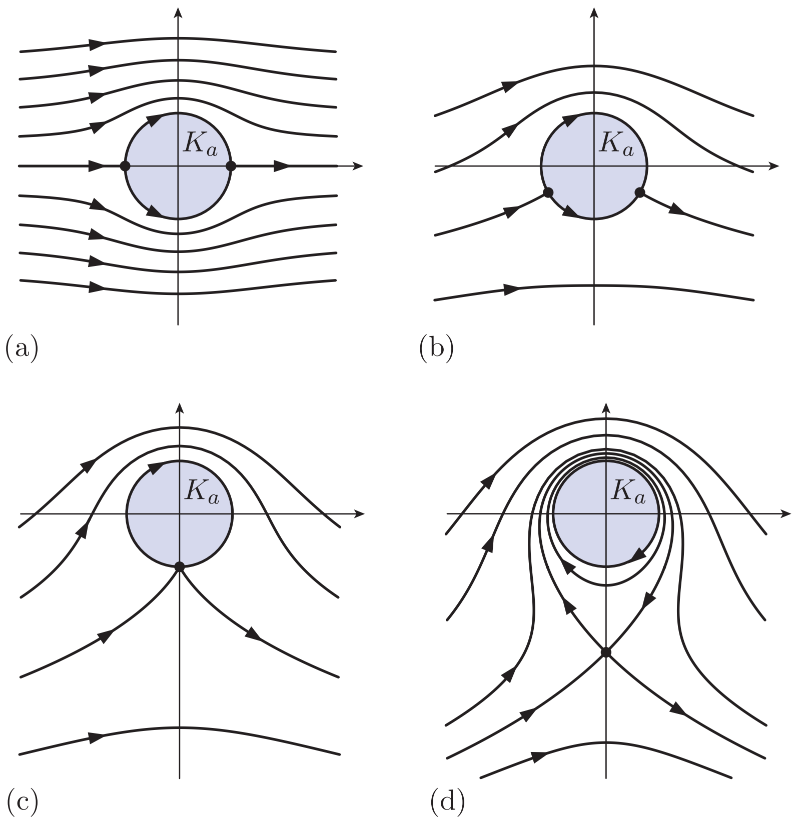


Figure 4.3 A closed disc K_a in a uniform stream with negative circulation $2\pi c$ when (a) $c = 0$, (b) $-2a < c < 0$, (c) $c = -2a$, (d) $c < -2a$

Since circulation is calculated using anticlockwise contours, the negative values of c in Figure 4.3 give rise to positive fluid flow clockwise around any simple-closed contour that surrounds K_a.

The shapes of the streamlines in Figure 4.3 are closely related to the locations of the stagnation points, shown by solid dots, in each case. For $c = 0$ (Figure 4.3(a)), the two stagnation points are symmetrically placed on the real axis at $z = \pm a$, by Exercise 4.1. As c decreases from 0 to $-2a$, the two stagnation points

$$z = \tfrac{1}{2}(ic \pm \sqrt{4a^2 - c^2}),$$

obtained in Exercise 4.2, move around the circle $\{z : |z| = a\}$, remaining symmetric with respect to the imaginary axis (Figure 4.3(b)), until $c = -2a$, when they coalesce to form a single stagnation point at $z = -ia$ (Figure 4.3(c)). For $c < -2a$, the single stagnation point moves off the circle down the imaginary axis (Figure 4.3(d)) at

$$z = \tfrac{1}{2}i(c - \sqrt{c^2 - 4a^2}).$$

In this case, the other solution of the equation $q_{a,c}(z) = 0$ lies inside the open disc $\{z : |z| < a\}$, as shown in the solution to Exercise 4.2(c).

A similar analysis can be carried out for positive values of c, resulting in streamline diagrams that are reflections in the real axis of the diagrams in Figure 4.3.

4.2 Obstacle Problem

We now discuss the problem of determining the flow past a more general obstacle in a uniform stream. To begin with, we make precise what we mean by an obstacle: this should be thought of as the two-dimensional cross-section of an object that extends indefinitely far perpendicular to the complex plane. Recall that the words *connected* and *compact* were defined in Subsections 4.3 and 5.1 of Unit A3, respectively.

Definition

An **obstacle** is a compact, connected set K in $\mathbb{C}$, which is not a single point, such that $\mathbb{C} - K$ is also connected.

Note that the set $\mathbb{C} - K$ is an unbounded region that surrounds K.

According to this definition, any closed disc is an obstacle, in particular

$$K_a = \{z : |z| \leq a\}$$

is an obstacle, as is any ellipse with its inside filled in. Also, any closed line segment

$$L(\alpha, \beta), \quad \text{where } \alpha \neq \beta,$$

is an obstacle.

The Obstacle Problem is to find, for a given obstacle K and a given $c \in \mathbb{R}$, a velocity function for an ideal flow which, roughly speaking, satisfies properties (a), (b) and (c) of Theorem 4.1. Properties (a) and (c) are easy to state for a general obstacle, but property (b) is more tricky. We want ∂K, the boundary of K, to look, as far as possible, as if it is made up of streamlines, but it is difficult to express this in terms of the velocity function q, because q can behave badly near some points, such as corners of ∂K.

Instead we express this streamline condition in terms of a stream function, which is the imaginary part of a complex potential function Ω for q. As with the complex potential function $\Omega_{a,c}$ used in Theorem 4.1, we cannot expect that Ω will be analytic on the whole of $\mathbb{C} - K$, but we try to make its region of analyticity as large as possible, so that we can give meaning to the streamline condition

$$\operatorname{Im} \Omega(z) = \text{constant}, \quad \text{for } z \in \partial K,$$

even though the whole of ∂K may not lie in the domain of Ω.

To make this idea precise in our statement of the Obstacle Problem, we use a simple path joining K to ∞ to play the role of a generalised cut, similar to the way we defined a cut plane in Subsection 5.1 of Unit C1. Recall from Subsection 1.1 of Unit B2 that a path is called *simple* if it does not intersect itself.

Obstacle Problem

Given an obstacle K and a real number c, we seek a velocity function q for an ideal flow with flow region $\mathcal{R} = \mathbb{C} - K$ satisfying the following properties.

(a) $\lim_{z\to\infty} q(z) = 1.$

(b) There is a complex potential function Ω for q on either $\mathcal{R}$ or $\mathcal{R} - \Sigma$, where Σ is a simple smooth path in $\mathcal{R}$ joining a point of K to ∞, and a real constant k such that

$$\lim_{z\to\alpha} \operatorname{Im}\Omega(z) = k, \quad \text{for each } \alpha \in \partial K.$$

(c) For any simple-closed contour Γ surrounding K,

$$\mathcal{C}_\Gamma = 2\pi c.$$

The quantity $2\pi c$ in property (c) is called the **circulation around the obstacle K**.

Remarks

1. Property (b) is illustrated in Figure 4.4 for the case where there is a complex potential Ω for q on $\mathcal{R} - \Sigma$.

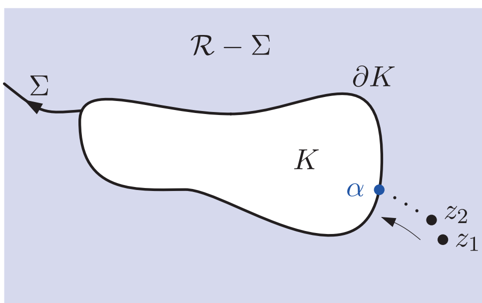


Figure 4.4 A smooth path Σ joining a point on the boundary of K to ∞

Recall that $\lim_{z\to\alpha} \operatorname{Im}\Omega(z) = k$ means that α is a limit point of $\mathcal{R} - \Sigma$, and for each sequence (z_n) in $\mathcal{R} - \Sigma$ such that $z_n \to \alpha$, we have $\operatorname{Im}\Omega(z_n) \to k$. Property (b) could be expressed equivalently by saying that the function $\operatorname{Im}\Omega$ has a continuous extension from $\mathcal{R} - \Sigma$ to $(\mathcal{R} - \Sigma) \cup \partial K$ with

$$\operatorname{Im}\Omega(z) = k, \quad \text{for } z \in \partial K.$$

2. Theorem 4.1 shows that the function

$$q_{a,c}(z) = \overline{1 - \frac{a^2}{z^2} - \frac{ic}{z}} \quad (z \in \mathbb{C} - \{0\})$$

solves the Obstacle Problem for $K = K_a = \{z : |z| \leq a\}$ with circulation $2\pi c$ around K. The simple smooth path Σ in property (b) is the part of the negative real axis which lies in $\mathbb{C} - K_a$ (see Figure 4.5), and the constant k is $-c \log a$, as in the proof of Theorem 4.1.

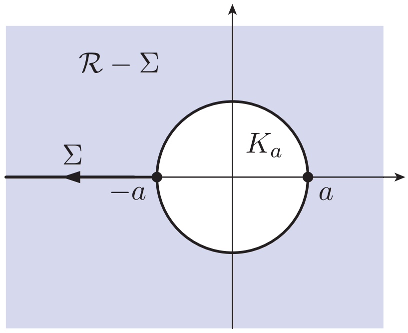


Figure 4.5 The path Σ for a circular obstacle

Note that if $c = 0$, then the simple smooth path Σ is not needed since the complex potential function $\Omega(z) = z + a^2/z$ is analytic on $\mathbb{C} - \{0\}$ and hence on $\mathbb{C} - K_a$.

3. We will refer to flow velocities *on* ∂K when it makes sense to do so. This will be the case when q has a continuous extension from $\mathbb{C} - K$ to ∂K (as $q_{a,c}$ does when $K = K_a$). Also, when this notion is ambiguous (for example, when the obstacle is a line segment with different limiting velocities as it is approached from either side) we may refer to velocities 'on either side of the obstacle', as we do in Remark 2 after Example 4.1 in Subsection 4.3.

4. Property (c) does not include the requirement that $\mathcal{F}_\Gamma = 0$. In fact, it is a consequence of property (b) that $\mathcal{F}_\Gamma = 0$, so such a requirement is automatically satisfied.

In the next exercise we ask you to verify that the velocity function $q(z) = 1$ solves the Obstacle Problem in a simple case with zero circulation (so the simple smooth path Σ is not needed).

Exercise 4.3

Show that the velocity function

$$q(z) = 1 \quad (z \in \mathbb{C} - K)$$

solves the Obstacle Problem for *any* closed line segment K parallel to the real axis with zero circulation around K.

4.3 Flow Mapping Theorem

Exercise 4.3 shows that the Obstacle Problem has a simple solution for an obstacle K consisting of a line segment parallel to the real axis when the circulation around K is 0. However, if the line segment is not parallel to the real axis, or if the circulation is not 0, then the Obstacle Problem is harder to solve.

To solve the Obstacle Problem for other obstacles and given circulations, we will use a technique based on conformal mappings. The idea is that if K is a given obstacle and f is a suitable conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_a$, where $K_a = \{z : |z| \leq a\}$, then it is possible to use our solution $q_{a,c}$ of the Obstacle Problem for K_a, found in Theorem 4.1, to solve the Obstacle Problem for the obstacle K. The details are given in the following result, the proof of which appears at the end of this section.

Theorem 4.2 Flow Mapping Theorem

Let K be an obstacle, and let f be a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_a$, where $a > 0$, such that the Laurent series about 0 for f on $\{z : |z| > R\}$ has the form

$$f(z) = z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots, \quad \text{for } |z| > R,$$

where $R > 0$ and $a_0, a_{-1}, a_{-2}, \ldots \in \mathbb{C}$. Then the velocity function

$$q(z) = q_{a,c}(f(z))\overline{f'(z)} \quad (z \in \mathbb{C} - K)$$

is the unique solution to the Obstacle Problem for K with circulation $2\pi c$ around K, and a corresponding complex potential function is

$$\Omega(z) = \Omega_{a,c}(f(z)).$$

Theorem 4.2 is illustrated in Figure 4.6, which also indicates that the conformal mapping f maps streamlines around K to streamlines around K_a. This is the case because $\Omega = \Omega_{a,c} \circ f$ and so, for any real constant k and any $z \in \mathbb{C} - K$,

$$\operatorname{Im} \Omega(z) = k \iff \operatorname{Im} \Omega_{a,c}(f(z)) = k.$$

Similarly, the conformal mapping f^{-1} maps streamlines around K_a to streamlines around K.

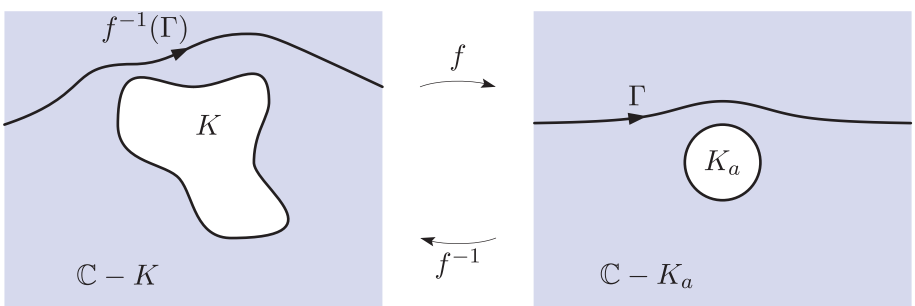


Figure 4.6 Streamlines around K are mapped to streamlines around K_a

Theorem 4.2 reduces the Obstacle Problem for K to the problem of finding a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_a$, for some $a > 0$. In general, it is a hard problem to find such mappings but there are some useful cases that can be solved using an explicit conformal mapping. Exercise 4.4 gives a straightforward example using the fact that any translation is a conformal mapping.

Exercise 4.4

Use the Flow Mapping Theorem to solve the Obstacle Problem for the closed disc

$$K = \{z : |z - \beta| \leq a\},$$

where $\beta \in \mathbb{C}$ and $a > 0$, with circulation $2\pi c$ around K.

To see more interesting solutions to the Obstacle Problem, we will use the inverse functions of the Joukowski functions introduced in Section 3. For example, in Theorem 3.1 you saw that the basic Joukowski function

$$J(z) = z + \frac{1}{z} \quad (z \in \mathbb{C} - \{0\})$$

gives a one-to-one conformal mapping from $\mathbb{C} - K_1$ onto $\mathbb{C} - K$, where $K_1 = \{z : |z| \leq 1\}$ and K is the line segment $L(-2, 2) = [-2, 2]$, and that $J(\partial K_1) = [-2, 2]$; see Figure 4.7.

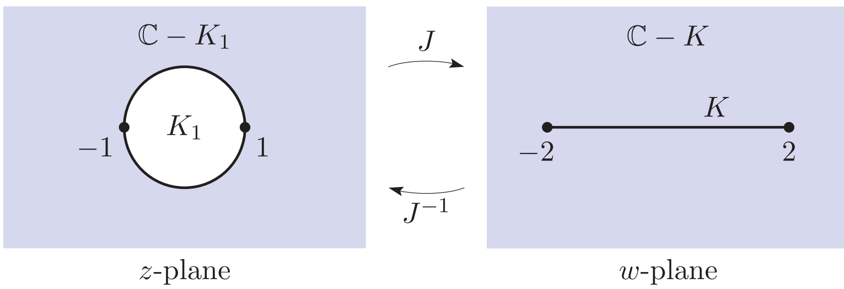


Figure 4.7 The geometric effect of the Joukowski function J

By Theorem 3.1(c), the restriction of J to $\mathbb{C} - K_1$ has inverse function

$$J^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4/w^2}\right) \quad (w \in \mathbb{C} - K).$$

It follows that if we *reverse* the roles of z and w here, as in Figure 4.8, then we can use J^{-1} as the function f in the Flow Mapping Theorem. We will then be able to use the Flow Mapping Theorem to solve the Obstacle Problem for the line segment $K = [-2, 2]$ with *non-zero* circulation around K.

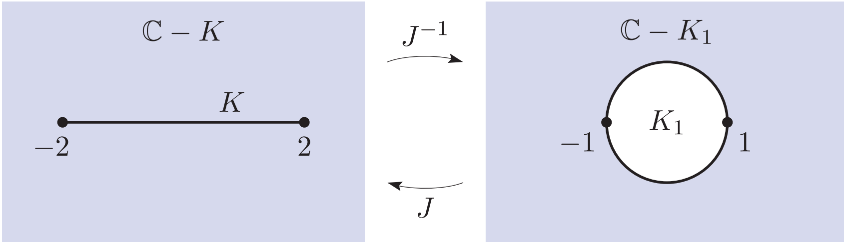


Figure 4.8 The geometric effect of the inverse function J^{-1}

The resulting formula for the velocity function q, obtained from Theorem 4.2, will involve the derivative of J^{-1}, which is rather complicated. However, there are several useful identities that can help simplify formulas involving inverse functions of Joukowski functions. These identities are given in Lemma 4.1, where they are stated for the general Joukowski function J_α, $\alpha \neq 0$, introduced in Subsection 3.2, which we will use later in the section.

From now on we will usually reverse the roles of the variables z and w when using Joukowski functions and their inverse functions.

Lemma 4.1

Let $J_\alpha(w) = w + \alpha^2/w$, where $\alpha \neq 0$, and let J_α^{-1} be the inverse function of the restriction of J_α to $\mathbb{C} - K_{|\alpha|}$. Then, for $z \in \mathbb{C} - L(-2\alpha, 2\alpha)$,

(a) $J_\alpha^{-1}(z) + \alpha^2/J_\alpha^{-1}(z) = z$

(b) $J_\alpha^{-1}(z) - \alpha^2/J_\alpha^{-1}(z) = z\sqrt{1 - 4\alpha^2/z^2}$

(c) $(J_\alpha^{-1})'(z) = \dfrac{1}{1 - \alpha^2/(J_\alpha^{-1}(z))^2} = \dfrac{J_\alpha^{-1}(z)}{z\sqrt{1 - 4\alpha^2/z^2}}.$

Proof Let $w = J_\alpha^{-1}(z)$. Part (a) holds because

$$z = J_\alpha(w) = w + \alpha^2/w = J_\alpha^{-1}(z) + \alpha^2/J_\alpha^{-1}(z).$$

Part (b) follows from part (a) together with the formula

$$J_\alpha^{-1}(z) = \tfrac{1}{2}\left(z + z\sqrt{1 - 4\alpha^2/z^2}\right),$$

given in Theorem 3.3(c).

For part (c), since $J_\alpha(w) = w + \alpha^2/w$, we have

$$J_\alpha'(w) = 1 - \alpha^2/w^2.$$

Thus, by the Inverse Function Rule (Theorem 3.2 of Unit A4), we obtain

$$(J_\alpha^{-1})'(z) = \frac{1}{J_\alpha'(w)} = \frac{1}{1 - \alpha^2/w^2} = \frac{1}{1 - \alpha^2/(J_\alpha^{-1}(z))^2}.$$

Hence, by part (b), we have

$$(J_\alpha^{-1})'(z) = \frac{J_\alpha^{-1}(z)}{J_\alpha^{-1}(z) - \alpha^2/J_\alpha^{-1}(z)} = \frac{J_\alpha^{-1}(z)}{z\sqrt{1 - 4\alpha^2/z^2}}.$$

■

Now we are able to solve the Obstacle Problem for the line segment $K = [-2, 2]$ with non-zero circulation around K.

Example 4.1

Use the function J^{-1} to show that the solution to the Obstacle Problem for $K = [-2, 2]$ with circulation $2\pi c$ around K is

$$q(z) = 1 - \overline{\left(\frac{ic}{z\sqrt{1-4/z^2}}\right)} \quad (z \in \mathbb{C} - K).$$

Solution

As pointed out above,

$$f(z) = J^{-1}(z) = \tfrac{1}{2}(z + z\sqrt{1-4/z^2})$$

is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_1$, so we try to apply the Flow Mapping Theorem with $a = 1$.

To verify the Laurent series condition, we need to express J^{-1} as a Laurent series about 0, valid on the exterior of some closed disc. To do this, we use the binomial series (Theorem 3.3 of Unit B3)

$$\begin{aligned}(1-4/z^2)^{1/2} &= 1 + \frac{1}{2}\left(-\frac{4}{z^2}\right) + \frac{\frac{1}{2}\times\left(-\frac{1}{2}\right)}{2!}\left(-\frac{4}{z^2}\right)^2 + \cdots \\ &= 1 - \frac{2}{z^2} - \frac{2}{z^4} - \cdots, \quad \text{for } |z| > 2,\end{aligned}$$

to give

$$\begin{aligned}J^{-1}(z) &= \frac{1}{2}\left(z + z\left(1 - \frac{2}{z^2} - \frac{2}{z^4} - \cdots\right)\right) \\ &= z - \frac{1}{z} - \frac{1}{z^3} - \cdots, \quad \text{for } |z| > 2.\end{aligned}$$

Thus J^{-1} is of the form given in the Laurent series condition.

It follows, by the Flow Mapping Theorem with $a = 1$, that the velocity function

$$\begin{aligned}q(z) &= q_{1,c}(J^{-1}(z))\overline{(J^{-1})'(z)} \\ &= \overline{\left(1 - \frac{1}{(J^{-1}(z))^2} - \frac{ic}{J^{-1}(z)}\right)}\,\overline{(J^{-1})'(z)}\end{aligned}$$

solves the Obstacle Problem for K with circulation $2\pi c$ around K.

We can simplify this formula by using the identities in Lemma 4.1 with $\alpha = 1$. First we use Lemma 4.1(c) to give

$$\begin{aligned}q(z) &= \overline{\left(1 - \frac{1}{(J^{-1}(z))^2} - \frac{ic}{J^{-1}(z)}\right)}\,\overline{\left(\frac{1}{1 - 1/(J^{-1}(z))^2}\right)} \\ &= 1 - \overline{\left(\frac{ic}{J^{-1}(z)(1 - 1/(J^{-1}(z))^2)}\right)},\end{aligned}$$

since

$$\left(1 - \frac{1}{(J^{-1}(z))^2}\right)\left(\frac{1}{1 - 1/(J^{-1}(z))^2}\right) = 1.$$

Then, by rearranging and using Lemma 4.1(b), we obtain

$$\begin{aligned} q(z) &= 1 - \overline{\left(\frac{ic}{J^{-1}(z) - 1/J^{-1}(z)}\right)} \\ &= 1 - \overline{\left(\frac{ic}{z\sqrt{1 - 4/z^2}}\right)}, \end{aligned}$$

as required.

Remarks

1. Notice that if $c = 0$ in Example 4.1, then we obtain the solution $q(z) = 1$ found in Exercise 4.3, as expected.
2. We can use the solution to Example 4.1 to compute the velocities on either side of the obstacle $K = [-2, 2]$ by putting $z = x + iy$ in the expression for $q(z)$ and letting y tend to 0 through positive values (written $y \to 0^+$) and through negative values (written $y \to 0^-$).

   For example, put $x = 0$, so $z = iy$. Then

$$\begin{aligned} z\sqrt{1 - 4/z^2} &= \begin{cases} i|y|\sqrt{1 - 4/(-y^2)}, & y > 0, \\ -i|y|\sqrt{1 - 4/(-y^2)}, & y < 0, \end{cases} \\ &= \begin{cases} i\sqrt{y^2 + 4}, & y > 0, \\ -i\sqrt{y^2 + 4}, & y < 0, \end{cases} \\ &\to \begin{cases} 2i & \text{as } y \to 0^+, \\ -2i & \text{as } y \to 0^-. \end{cases} \end{aligned}$$

   Thus, with $z = iy$ again,

$$q(iy) = 1 - \overline{\left(\frac{ic}{z\sqrt{1 - 4/z^2}}\right)} \to \begin{cases} 1 - \frac{1}{2}c & \text{as } y \to 0^+, \\ 1 + \frac{1}{2}c & \text{as } y \to 0^-. \end{cases} \qquad (4.2)$$

   These values show that if, for example, the circulation is negative ($c < 0$), then the flow speed above the obstacle is greater than the flow speed below it.
3. An alternative method to obtain the velocity function in Example 4.1 is to first determine the complex potential function $\Omega = \Omega_{a,c} \circ f$ (given by the Flow Mapping Theorem), then differentiate Ω and take its conjugate to give $q = \overline{\Omega'}$. We ask you to use this approach in the next exercise.

Exercise 4.5

Show that the solution of the Obstacle Problem for $K = [-2, 2]$ with circulation $2\pi c$ around K is

$$q(z) = 1 - \overline{\left(\frac{ic}{z\sqrt{1 - 4/z^2}}\right)} \quad (z \in \mathbb{C} - K),$$

by first finding a complex potential function Ω for this flow.

(*Hint*: The fact that $\Omega_{1,c}(w) = J(w) - ic \operatorname{Log} w$ will help, as will Lemma 4.1(c).)

The complex potential function Ω found in Exercise 4.5 should, in principle, enable us to plot the streamlines for the corresponding flow past $K = [-2, 2]$, since the streamlines are determined by equations of the form

$$\operatorname{Im} \Omega(z) = \text{constant}.$$

However, if $c \neq 0$, then this equation is complicated, and the streamlines can be plotted accurately only with a computer. You can get a rough idea of the shape of the streamlines around a given obstacle K by using the conformal mapping f from the Flow Mapping Theorem to transfer the streamline diagram from Figure 4.3 across from $\mathbb{C} - K_a$ to $\mathbb{C} - K$, paying attention to the location of the stagnation points. See Figure 4.9, for example.

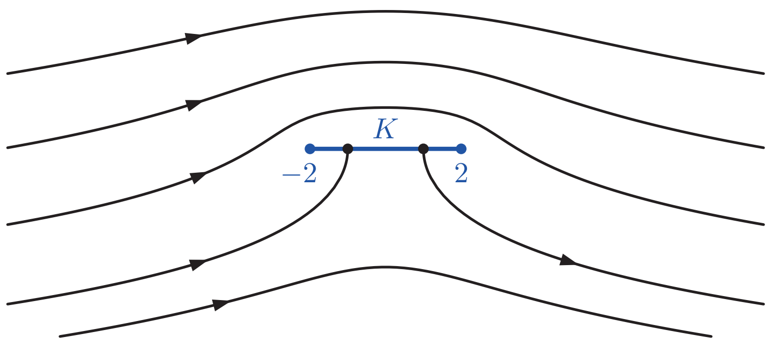


Figure 4.9 Streamlines near $K = [-2, 2]$ for a uniform flow with negative circulation around K, showing the stagnation points on K

To deal with somewhat more complicated obstacles, we will employ the more general family of Joukowski functions

$$J_a(z) = z + \frac{a^2}{z} \quad (z \in \mathbb{C} - \{0\}),$$

where $a > 0$, introduced in Subsection 3.2.

The results of Theorem 3.2 show that if we reverse the roles of z and w, then we can use J_a^{-1} as the function f in the Flow Mapping Theorem and therefore solve the Obstacle Problem for the obstacle $K = [-2a, 2a]$, where $a > 0$. The following exercise generalises Example 4.1, and the solution is similar.

Exercise 4.6

Use the function J_a^{-1}, and the identities in Lemma 4.1 with $\alpha = a$, where $a > 0$, to show that the solution to the Obstacle Problem for $K = [-2a, 2a]$ with circulation $2\pi c$ around K is

$$q(z) = 1 - \overline{\left(\frac{ic}{z\sqrt{1 - 4a^2/z^2}}\right)} \quad (z \in \mathbb{C} - K).$$

The functions J_a can also be used to solve the Obstacle Problem for flow past various other obstacles. In Exercise 3.2(a) we saw that if $r > a$, then J_a maps the circle $C_r = \{z : |z| = r\}$ onto the ellipse in the w-plane shown in Figure 4.10.

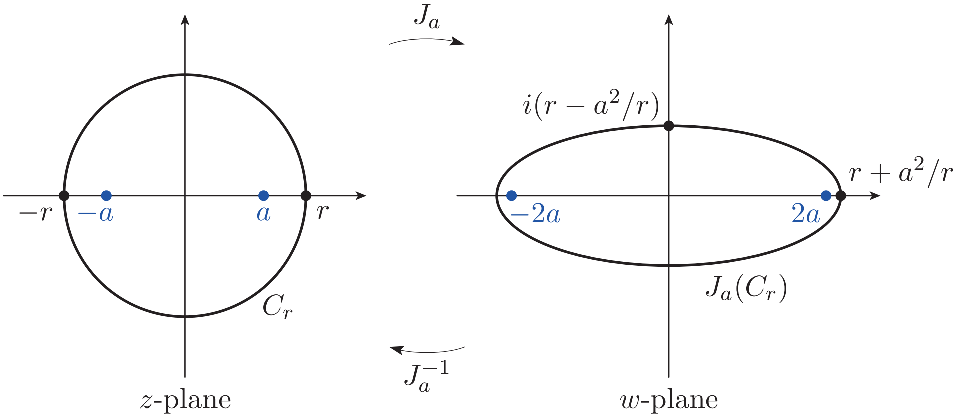


Figure 4.10 A Joukowski function mapping a circle onto an ellipse

In the next exercise you are asked to use this result to solve the Obstacle Problem for flow past this ellipse. Once again the roles of z and w have to be reversed from those in Figure 4.10.

Exercise 4.7

Let K be the obstacle that comprises the ellipse shown on the right-hand side of Figure 4.10 with its inside filled in. Use the Flow Mapping Theorem to show that the solution to the Obstacle Problem for K with circulation $2\pi c$ around K is

$$q(z) = 1 - \overline{\left(\frac{r^2 - a^2 + icJ_a^{-1}(z)}{(J_a^{-1}(z))^2 - a^2}\right)} \quad (z \in \mathbb{C} - K).$$

(*Hint*: The solution is similar to the solution to Exercise 4.6, especially checking the Laurent series condition, but the algebra in calculating the formula for $q(z)$ is slightly more complicated.)

In Exercise 4.7 you were asked to find the velocity function $q(z)$ expressed in terms of $J_a^{-1}(z)$ rather than in terms of z since substituting the formula for $J_a^{-1}(z)$ leads to a complicated formula for the velocity function here.

4.4 Proof of the Flow Mapping Theorem

We end this section with a proof of the Flow Mapping Theorem. This proof may be omitted on a first reading.

Theorem 4.2 Flow Mapping Theorem

Let K be an obstacle, and let f be a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_a$, where $a > 0$, such that the Laurent series about 0 for f on $\{z : |z| > R\}$ has the form

$$f(z) = z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots, \quad \text{for } |z| > R,$$

where $R > 0$ and $a_0, a_{-1}, a_{-2}, \ldots \in \mathbb{C}$. Then the velocity function

$$q(z) = q_{a,c}(f(z))\overline{f'(z)} \quad (z \in \mathbb{C} - K)$$

is the unique solution to the Obstacle Problem for K with circulation $2\pi c$ around K, and a corresponding complex potential function is

$$\Omega(z) = \Omega_{a,c}(f(z)).$$

Proof We must show that properties (a), (b) and (c) of the Obstacle Problem are satisfied. First we observe that

$$\begin{aligned} q(z) &= q_{a,c}(f(z))\overline{f'(z)} \\ &= \overline{\left(1 - \frac{a^2}{(f(z))^2} - \frac{ic}{f(z)}\right) f'(z)} \end{aligned}$$

is a velocity function for an ideal flow on $\mathcal{R} = \mathbb{C} - K$, since it is the conjugate of an analytic function there.

Now, using the Laurent series for f, it can be shown that

$$\lim_{z\to\infty} \frac{1}{f(z)} = 0 \quad \text{and} \quad \lim_{z\to\infty} f'(z) = 1.$$

It follows that

$$\lim_{z\to\infty} q(z) = \lim_{z\to\infty} \overline{\left(1 - \frac{a^2}{(f(z))^2} - \frac{ic}{f(z)}\right) f'(z)} = 1,$$

which verifies property (a) of the Obstacle Problem.

To verify property (b) of the Obstacle Problem, we use the fact that f is one-to-one and analytic on $\mathcal{R}$ so, by the Inverse Function Rule (Theorem 3.3 of Unit C2), f has an inverse function f^{-1} which is analytic on $\mathbb{C} - K_a$. Since $\Omega_{a,c}$ is analytic on the cut plane $\mathbb{C}_\pi$, it follows that the composite function $\Omega_{a,c} \circ f$ is analytic on the region

$$f^{-1}(\mathbb{C}_\pi - K_a) = \mathcal{R} - f^{-1}(\Sigma_a),$$

where $\Sigma_a = \{u \in \mathbb{R} : u < -a\}$ is the part of the negative real axis that lies in $\mathbb{C} - K_a$ (see Figure 4.11).

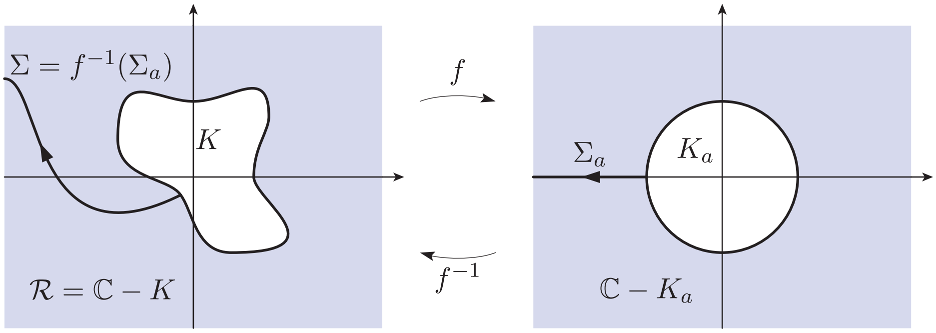


Figure 4.11 Finding a simple smooth path joining K to ∞

Since Σ_a is a simple smooth path joining K_a to ∞, and f^{-1} is one-to-one and conformal, it follows that $\Sigma = f^{-1}(\Sigma_a)$ is a simple smooth path joining K to ∞. Moreover, $\Omega = \Omega_{a,c} \circ f$ is a complex potential function for q on $\mathcal{R} - \Sigma$, because, by the Chain Rule,

$$\begin{aligned}\overline{\Omega'(z)} &= \overline{\Omega'_{a,c}(f(z))f'(z)}\\ &= \overline{\left(1 - \frac{a^2}{(f(z))^2} - \frac{ic}{f(z)}\right)f'(z)}\\ &= q(z), \quad \text{for } z \in \mathcal{R} - \Sigma.\end{aligned}$$

To complete the verification of property (b), we need to show that there is a constant k such that

$$\lim_{z \to \alpha} \operatorname{Im} \Omega(z) = k, \quad \text{for each } \alpha \in \partial K. \tag{4.3}$$

To do this, first note that if $\alpha \in \partial K$, then α is a limit point of $\mathcal{R} - \Sigma$; indeed, each open disc with centre α must contain a point of $\mathcal{R} - \Sigma$, so we can construct a sequence (z_n) in $\mathcal{R} - \Sigma$ such that $z_n \to \alpha$.

Now suppose that (z_n) lies in $\mathcal{R} - \Sigma$, and $z_n \to \alpha$. Then the sequence $(w_n) = (f(z_n))$ lies in $\mathbb{C}_\pi - K_a$, and we claim that $|w_n| \to a$. Indeed, if $\varepsilon > 0$ is given and $\Gamma_\varepsilon = \{w : |w| = a + \varepsilon\}$, then $f^{-1}(\Gamma_\varepsilon)$ is a simple-closed contour in $\mathcal{R}$ which surrounds but does not meet K (see Figure 4.12). Moreover, f^{-1} maps the outside of Γ_ε onto the outside of $f^{-1}(\Gamma_\varepsilon)$. These plausible assertions can be verified by using the fact that f is a one-to-one conformal mapping of the form given by the Laurent series condition; we omit the details.

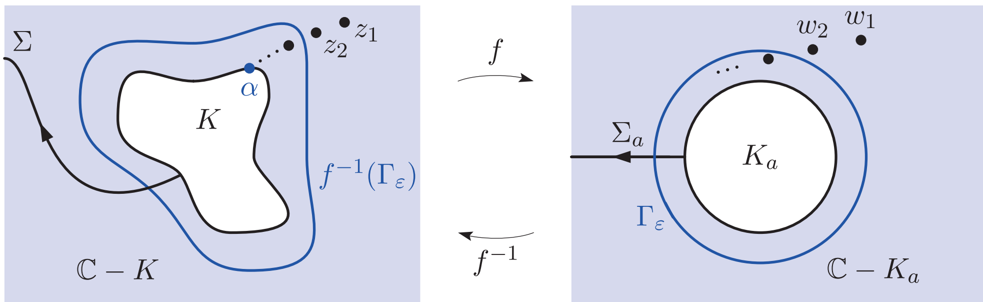


Figure 4.12 Finding the value of $\operatorname{Im} \Omega(z)$ on ∂K

It follows that the terms of (z_n) must eventually lie inside $f^{-1}(\Gamma_\varepsilon)$, and hence that the terms of (w_n) must eventually lie between Γ_ε and ∂K_a. Hence $|w_n| \to a$ as $n \to \infty$.

Therefore, if $w_n = u_n + iv_n$, for $n = 1, 2, \ldots$, then we can use the formula

$$\Omega_{a,c}(z) = z + \frac{a^2}{z} - ic\,\mathrm{Log}\,z$$

from Theorem 4.1 to see that

$$\begin{aligned}\mathrm{Im}\,\Omega(z_n) &= \mathrm{Im}\,\Omega_{a,c}(f(z_n)) \\ &= \mathrm{Im}\,\Omega_{a,c}(w_n) \\ &= v_n\left(1 - \frac{a^2}{|w_n|^2}\right) - c\log|w_n| \\ &\to -c\log a \text{ as } n \to \infty.\end{aligned}$$

Thus condition (4.3) holds with $k = -c\log a$.

Finally, we check property (c). If Γ is any simple-closed contour surrounding K, then $f(\Gamma)$ is a simple-closed contour surrounding K_a and so

$$\begin{aligned}\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma &= \int_\Gamma \overline{q}(z)\,dz \\ &= \int_\Gamma \overline{q_{a,c}}(f(z))f'(z)\,dz \\ &= \int_{f(\Gamma)} \overline{q_{a,c}}(w)\,dw = 2\pi c \quad \text{(by Theorem 4.1(c)).}\end{aligned}$$

The validity of the substitution used here, $w = f(z)$, $dw = f'(z)\,dz$, can be justified as in the proof of the Argument Principle (Theorem 2.3 of Unit C2).

Hence $\mathcal{C}_\Gamma = 2\pi c$, as required (and also $\mathcal{F}_\Gamma = 0$). ■

Remark

We have omitted the proof of the *uniqueness* of the solution q, which involves a slightly complicated application of the Maximum Principle (Theorem 4.2 of Unit C2).

Further exercises

Exercise 4.8

By considering the stream function of the flow with complex potential

$$\Omega(z) = z + \frac{a^2}{z} \quad (z \in \mathbb{C} - K_a),$$

where $K_a = \{z : |z| \le a\}$, find the equation of the streamline which passes through the point $2ia$. Hence describe the behaviour of this streamline far from the disc K_a.

5 Flow past an aerofoil

After working through this section, you should be able to:

- explain what is meant by a *Joukowski aerofoil*, its *trailing edge* and its *angle of attack*
- solve the Obstacle Problem for flows around an aerofoil, with or without circulation, by applying the Flow Mapping Theorem.

5.1 Aerofoils

In Subsection 3.2 we saw that if $a, b > 0$, then the Joukowski function J_a is a one-to-one mapping of the circle $C = \{w : |w + b| = a + b\}$ onto an aerofoil shaped curve with a cusp at $z = 2a$. The obstacle with boundary $J_a(c)$ is an example of a *Joukowski aerofoil* (see Figure 5.1).

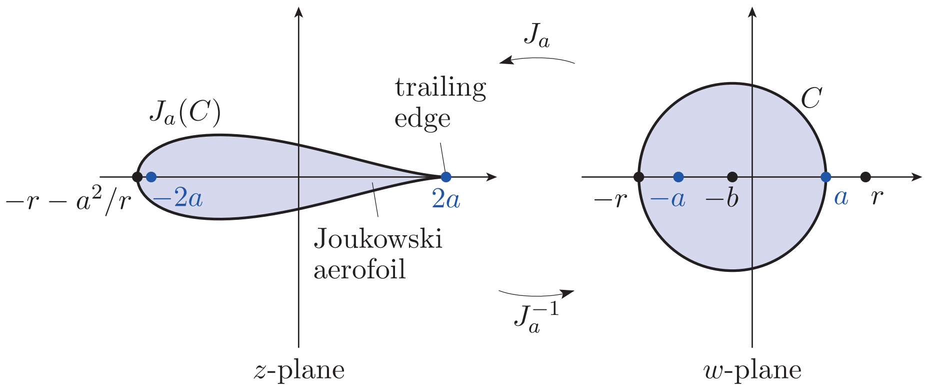


Figure 5.1 J_a^{-1} maps an aerofoil-shaped curve onto a circle

Observe that J_a has critical points a and $-a$, since $J_a'(w) = 0$ if and only if $w = \pm a$, and C passes through the point a and surrounds the point $-a$.

Definitions

A **Joukowski aerofoil** is an obstacle that has boundary $J_a(C)$, for $a > 0$ (possibly after an appropriate translation or rotation), where C is a circle that passes through one of the critical points $w = a$ of J_a and surrounds the other critical point $w = -a$.

The point $z = 2a$ is called the **trailing edge** of the aerofoil.

Remarks

1. The boundary $J_a(C)$ of an aerofoil is smooth except at the trailing edge $z = 2a$, where there is a cusp.
2. In order for a Joukowski aerofoil to be symmetric under reflection in the central axis through the trailing edge, the centre of the circle C must lie on the (negative) real axis, as in Figure 5.1.

The next exercise leads you in stages through the process of using the conformal mapping J_a^{-1} to find the velocity function for the flow past the aerofoil in Figure 5.1, and then use the formula for this velocity function to suggest that in real fluid flow past this aerofoil the circulation would have to be 0.

Exercise 5.1

Suppose that C is the circle $\{w : |w+b| = a+b\}$, where $a, b > 0$, and that K is the Joukowski aerofoil for which $\partial K = J_a(C)$.

(a) Write down a conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_{a+b}$.

(b) Deduce from part (a) that the solution to the Obstacle Problem for K with circulation $2\pi c$ around K corresponds to the complex potential function

$$\Omega(z) = w + b + \frac{(a+b)^2}{w+b} - ic\,\mathrm{Log}(w+b),$$

where $w = J_a^{-1}(z)$.

(c) Show that the velocity function q for this flow is

$$q(z) = \overline{\frac{w^2}{w^2-a^2}\left(1 - \frac{(a+b)^2}{(w+b)^2} - \frac{ic}{w+b}\right)} \quad (z \in \mathbb{C} - K),$$

where $w = J_a^{-1}(z)$.

(d) Show that $\lim\limits_{z\to 2a} q(z)$ (the limiting velocity at the trailing edge) exists if and only if $c = 0$.

(*Hint*: The identity

$$1 - \frac{(a+b)^2}{(w+b)^2} = \frac{(w+b)^2-(a+b)^2}{(w+b)^2} = \frac{(w+a+2b)(w-a)}{(w+b)^2}$$

helps solve this last part.)

5.2 Rotated obstacles

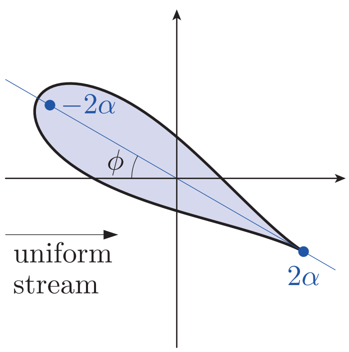


Figure 5.2 An aerofoil inclined at an angle to a uniform stream

Apart from one case involving translation (Exercise 4.4), we have so far considered examples of obstacles that are symmetric under reflection in the real axis. While an aeroplane is in flight, the wing cross-section is at a slight angle to the oncoming airstream. We can model this situation by placing a symmetric Joukowski aerofoil at an angle ϕ to a uniform stream, and attempting to find a complex potential for this version of the Obstacle Problem (see Figure 5.2). In these circumstances, ϕ is known as the aerofoil's **angle of attack**.

In order to make progress in dealing with obstacles that have been rotated in this way, we use the general Joukowski functions

$$J_\alpha(w) = w + \frac{\alpha^2}{w},$$

where $\alpha \in \mathbb{C} - \{0\}$, which were introduced in Subsection 3.2. The main properties of these functions were given in Theorem 3.3.

We recall also from Subsection 3.2 (see equation (3.1)) that if R_ϕ denotes the rotation about 0 through the angle $\phi = -\operatorname{Arg}\alpha$, then

$$J_\alpha = R_\phi^{-1} \circ J_{|\alpha|} \circ R_\phi.$$

This decomposition of J_α is illustrated in Figure 5.3, which shows that J_α maps $\mathbb{C} - K_{|\alpha|}$ onto the region that is the complement of the closed line segment $L(-2\alpha, 2\alpha)$ joining -2α and 2α, and it maps the circle $C_{|\alpha|}$ to $L(-2\alpha, 2\alpha)$ itself.

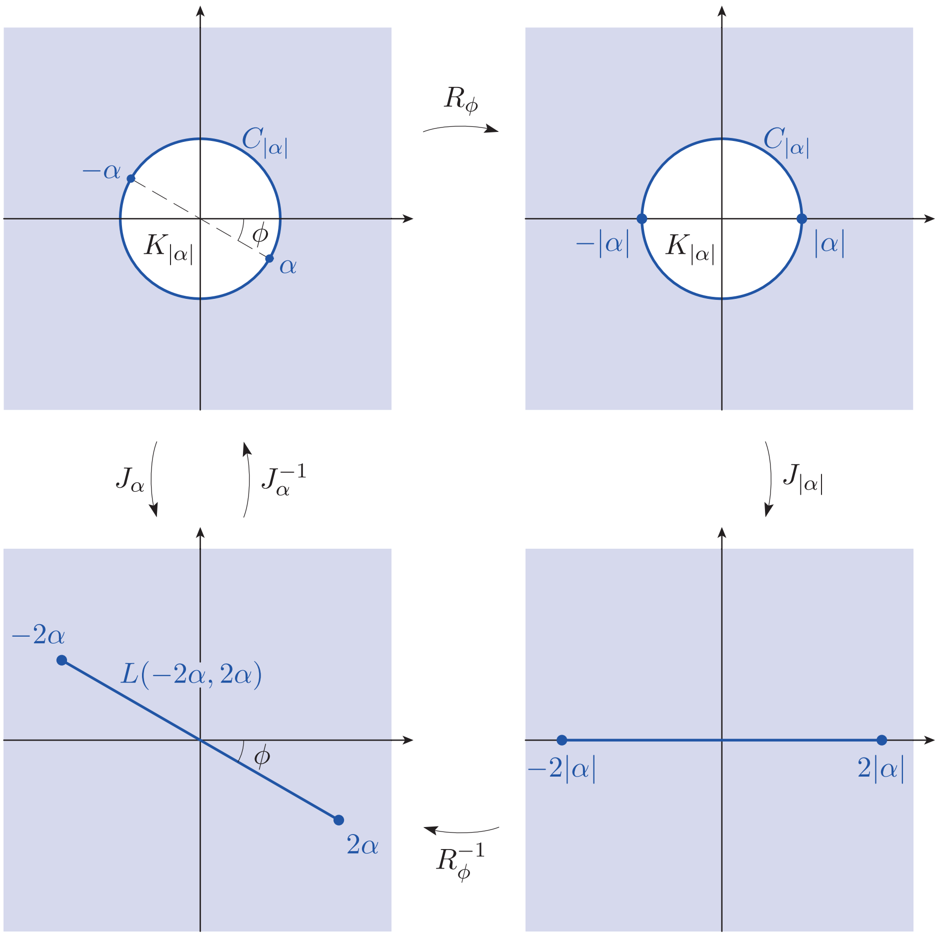


Figure 5.3 Decomposition of a Joukowski function J_α

It follows that the inverse function J_α^{-1} is a conformal mapping from $\mathbb{C} - L(-2\alpha, 2\alpha)$ onto $\mathbb{C} - K_{|\alpha|}$. This fact enables us to solve the Obstacle Problem for such a line segment in the following example.

Example 5.1

(a) Find a complex potential that provides a solution to the Obstacle Problem when the obstacle K is the line segment $L(-2\alpha, 2\alpha)$ and the circulation around K is $2\pi c$.

(b) Verify, from your answer to part (a) with $c = 0$, that the circulation-free complex potential for $\alpha = a$ (where a is real) is $\Omega(z) = z$, as expected.

(c) Determine the velocity function q which solves the Obstacle Problem for $K = L(-ia, ia)$ with zero circulation. (Here K can be regarded as a thin plate of width $2a$ placed at right angles to a uniform flow.)

Solution

(a) As noted above, J_α^{-1} is a one-to-one conformal mapping from $\mathbb{C} - L(-2\alpha, 2\alpha)$ onto $\mathbb{C} - K_{|\alpha|}$. Also, as in the solution to Exercise 4.6, with α in place of a, we have

$$\begin{aligned} J_\alpha^{-1}(z) &= \tfrac{1}{2}\left(z + z\sqrt{1 - 4\alpha^2/z^2}\right) \\ &= z - \frac{\alpha^2}{z} - \frac{\alpha^4}{z^3} - \cdots, \quad \text{for } |z| > 2|\alpha|, \end{aligned}$$

so the function $f = J_\alpha^{-1}$ satisfies the Laurent series condition of the Flow Mapping Theorem.

By the Flow Mapping Theorem, a suitable complex potential for the flow past K, with circulation $2\pi c$ around K, is

$$\begin{aligned} \Omega(z) &= (\Omega_{|\alpha|,c} \circ J_\alpha^{-1})(z) \\ &= J_\alpha^{-1}(z) + \frac{|\alpha|^2}{J_\alpha^{-1}(z)} - ic\,\mathrm{Log}(J_\alpha^{-1}(z)) \\ &= z + \frac{|\alpha|^2 - \alpha^2}{J_\alpha^{-1}(z)} - ic\,\mathrm{Log}(J_\alpha^{-1}(z)), \end{aligned}$$

since $J_\alpha^{-1}(z) + \dfrac{\alpha^2}{J_\alpha^{-1}(z)} = z$, by Lemma 4.1(a).

(b) With $c = 0$ we have

$$\Omega(z) = z + \frac{|\alpha|^2 - \alpha^2}{J_\alpha^{-1}(z)},$$

so if $\alpha = a$, where a is real, then the complex potential is

$$\Omega(z) = z,$$

as in the solution to Exercise 4.3.

(c) If $c = 0$ and $\alpha = \frac{1}{2}ia$, then, by part (a), we have

$$\begin{aligned}
\Omega(z) &= z + \frac{|\alpha|^2 - \alpha^2}{J_\alpha^{-1}(z)} \\
&= z + \frac{\frac{1}{4}a^2 - \left(-\frac{1}{4}a^2\right)}{\frac{1}{2}z(1+\sqrt{1+a^2/z^2})} \\
&= z + \frac{a^2(1-\sqrt{1+a^2/z^2})}{z(1+\sqrt{1+a^2/z^2})(1-\sqrt{1+a^2/z^2})} \\
&= z + \frac{a^2(1-\sqrt{1+a^2/z^2})}{-a^2/z} = z\sqrt{1+\frac{a^2}{z^2}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\Omega'(z) &= \sqrt{1+a^2/z^2} + z \times \frac{1}{2}\frac{1}{\sqrt{1+a^2/z^2}} \times \left(\frac{-2a^2}{z^3}\right) \\
&= \sqrt{1+a^2/z^2} - \frac{a^2}{z^2}\frac{1}{\sqrt{1+a^2/z^2}} \\
&= \frac{1+a^2/z^2 - a^2/z^2}{\sqrt{1+a^2/z^2}} \\
&= 1/\sqrt{1+a^2/z^2}.
\end{aligned}$$

Hence the required velocity function is

$$q(z) = \overline{\Omega'(z)} = \overline{1/\sqrt{1+a^2/z^2}}.$$

Remark

In part (c) the velocity function q could have been obtained directly from the formula $q(z) = q_{|\alpha|,c}\left(J_\alpha^{-1}(z)\right)\overline{\left(J_\alpha^{-1}\right)'(z)}$, as we did in the solution to Example 4.1. Also, note that the velocity function q in this example is unbounded near $\pm ia$, the endpoints of the line segment.

By now you should have started to become familiar with the strategy for solving the Obstacle Problem using the Flow Mapping Theorem. The key step is to find a conformal mapping f from the region outside the given obstacle K onto a region that is the outside of some closed disc $K_a = \{z : |z| \le a\}$, $a > 0$, with the property that

$$f(z) = z + a_0 + \frac{a_{-1}}{z} + \cdots, \quad \text{for } |z| > R,$$

where $R > 0$. You can then apply the Flow Mapping Theorem to find a complex potential Ω and corresponding complex velocity q, given by

$$\Omega(z) = \Omega_{a,c}(f(z)) = f(z) + \frac{a^2}{f(z)} - ic\,\mathrm{Log}\, f(z)$$

and

$$q(z) = q_{a,c}(f(z))\overline{f'(z)} = \overline{\left(1 - \frac{a^2}{(f(z))^2} - \frac{ic}{f(z)}\right)f'(z)} \quad (z \in \mathbb{C} - K),$$

which describe the flow past the given obstacle in a uniform stream with circulation $2\pi c$.

In general, there is no simple procedure for finding such a conformal mapping. In the examples you have seen so far, the conformal mapping f was usually an inverse Joukowski function, but explicit mappings can be found for a variety of other obstacles by using combinations and compositions of the following types of mappings:

- translations, that is, mappings of the form $z \longmapsto z + \beta$, where $\beta \neq 0$
- complex scalings, that is, mappings of the form $z \longmapsto \lambda z$, with $\lambda \neq 0$, and in particular rotations, when $|\lambda| = 1$
- Joukowski functions J_α, where $\alpha \neq 0$
- inverse Joukowski functions J_α^{-1}, where $\alpha \neq 0$.

The next two exercises ask you to use combinations or compositions of two such conformal mappings to solve the Obstacle Problem for more complicated obstacles.

Exercise 5.2

The Joukowski aerofoil K shown in the figure below is obtained by rotating the aerofoil in Exercise 5.1 (and Figure 5.1) clockwise about the origin through an angle ϕ, where $0 < \phi < \pi/2$. We then have

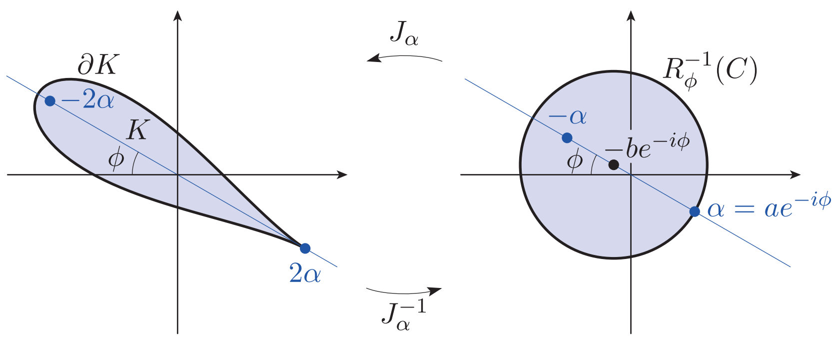
$$\partial K = (R_\phi^{-1} \circ J_a)(C),$$

where C is the circle $\{w : |w + b| = a + b\}$.

(a) With $\alpha = ae^{-i\phi}$, use the equation $J_\alpha = R_\phi^{-1} \circ J_{|\alpha|} \circ R_\phi$ to show that the aerofoil boundary ∂K is the image under J_α of the circle

$$R_\phi^{-1}(C) = \{w : |w + be^{-i\phi}| = a + b\},$$

as shown in the figure.



Hence write down a conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_{a+b}$.

(b) Show that a complex potential for the solution to the Obstacle Problem for K with circulation $2\pi c$ is

$$\Omega(z) = w + be^{-i\phi} + \frac{(a+b)^2}{w + be^{-i\phi}} - ic\,\mathrm{Log}(w + be^{-i\phi}),$$

where $w = J_\alpha^{-1}(z)$.

(c) Determine the corresponding velocity function q, in terms of $w = J_\alpha^{-1}(z)$.

Exercise 5.3

Let K_a denote the closed disc $\{z : |z| \leq a\}$. Consider the obstacle

$$K = [-2, -1] \cup K_1 \cup [1, 2],$$

consisting of a closed disc of radius 1 with two line segments attached, illustrated in Figure 5.4.

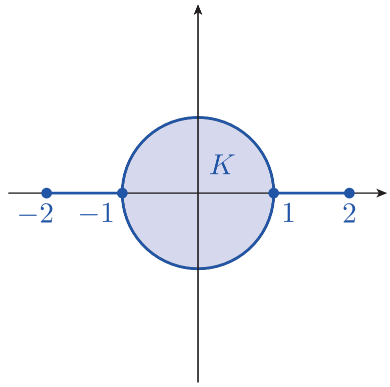


Figure 5.4 The obstacle K

In Exercise 3.4 you saw that the Joukowski function J is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - \left[-\frac{5}{2}, \frac{5}{2}\right]$.

(a) Find a value of a for which the function

$$J_a(w) = w + \frac{a^2}{w}$$

is a one-to-one conformal mapping from $\mathbb{C} - K_a$ onto $\mathbb{C} - \left[-\frac{5}{2}, \frac{5}{2}\right]$.

(b) Show that the Obstacle Problem for K with circulation $2\pi c$ around K has solution

$$q(z) = \overline{\left(1 - \frac{ic}{J(z)\sqrt{1 - (25/4)/(J(z))^2}}\right) J'(z)} \quad (z \in \mathbb{C} - K),$$

where J is the basic Joukowski function.

(*Hint*: You can assume that the function $f = J_a^{-1} \circ J$ satisfies the Laurent series condition of the Flow Mapping Theorem. Also, find the velocity function from the complex potential function by using the fact that $\Omega_{a,c} = J_a - ic\,\mathrm{Log}$ and the approach of Exercise 4.5.)

Further exercises

Exercise 5.4

Suppose that $0 < a < r$. Let K be the obstacle shown in Figure 5.5 whose boundary is the ellipse

$$\partial K = \left\{x + iy : \frac{x^2}{(r - a^2/r)^2} + \frac{y^2}{(r + a^2/r)^2} = 1\right\}.$$

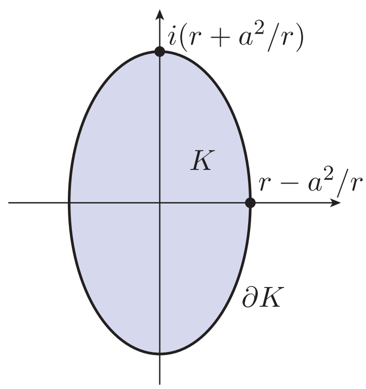


Figure 5.5 An ellipse with its major axis on the imaginary axis

Use the solution to Exercise 3.3 to show that the Obstacle Problem for K with circulation $2\pi c$ around K has solution

$$q(z) = 1 - \overline{\left(\frac{r^2 + a^2 + icJ_{ia}^{-1}(z)}{(J_{ia}^{-1}(z))^2 + a^2}\right)} \quad (z \in \mathbb{C} - K),$$

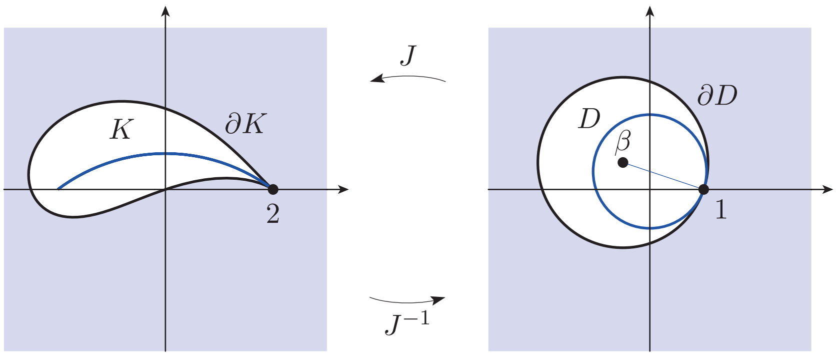
by first finding a suitable complex potential function for this flow.

Exercise 5.5

The asymmetric (or *cambered*) Joukowski aerofoil K, shown on the left of the figure below, has boundary $\partial K = J(\partial D)$, where

$$D = \{w : |w - \beta| \leq |1 - \beta|\}$$

and β is a complex number with $\operatorname{Re}\beta < 0$. Here J is the basic Joukowski function, and you can assume that J maps $\mathbb{C} - D$ conformally onto $\mathbb{C} - K$.



Show that the Obstacle Problem for K with circulation $2\pi c$ around K has solution

$$q(z) = \overline{\left(1 - \frac{|1-\beta|^2}{(w-\beta)^2} - \frac{ic}{w-\beta}\right)\frac{w^2}{w^2-1}} \qquad (z \in \mathbb{C} - K),$$

where $w = J^{-1}(z)$.

(The condition $\operatorname{Re}\beta < 0$ ensures that the critical point $w = -1$ of J lies inside ∂D, and its image $z = J(-1) = -2$ lies inside ∂K.

Note that if $\operatorname{Re}\beta = 0$ and $\operatorname{Im}\beta > 0$, then the circle ∂D passes through both of the critical points $w = \pm 1$ of J. As shown in Exercise 3.5, the image $J(\partial D)$ is a circular arc in the upper half-plane from $z = 2$ to $z = -2$ and back. The case $\operatorname{Re}\beta = 0$ and $\operatorname{Im}\beta > 0$ is shown by blue curves on the diagram.)

Why can aeroplanes fly?

Early in the twentieth century, Zhukovsky and the German mathematicians Martin Wilhelm Kutta (1867–1944) and Paul Richard Heinrich Blasius (1883–1970) each independently calculated the force on an obstacle K in a uniform stream in order to try to explain the upward force experienced by the wing of an aeroplane in flight. Here we describe this work in outline and briefly discuss the usefulness of the model of fluid flow given in this unit.

In an ideal fluid of constant density ρ, the pressure $p(z)$ and the velocity $q(z)$ within the flow region are related by **Bernoulli's equation**,

$$p(z) = p_0 - \tfrac{1}{2}\rho|q(z)|^2, \quad \text{where } p_0 \text{ is a constant.}$$

So fluid velocity of higher magnitude corresponds to lower pressure.

As an example, consider the case of a closed disc K_a, $a > 0$, in a uniform stream with circulation $2\pi c$, where the velocity is

$$q(z) = \overline{1 - \frac{a^2}{z^2} - \frac{ic}{z}}.$$

Substituting this formula for $q(z)$ into Bernoulli's equation and evaluating the pressure at a point $z = ae^{it}$, $t \in (-\pi, \pi]$, gives

$$p(ae^{it}) = p_0 - \tfrac{1}{2}\rho(2\sin t - c/a)^2.$$

Thus for negative circulation the pressure is greater on the lower half of ∂K_a, where $\sin t < 0$, than it is on the upper half, where $\sin t > 0$. This difference leads to an upward force on K_a.

The force F on a general obstacle K with smooth boundary ∂K due to pressure on the boundary can be calculated using the formula

$$F = i\int_0^L p(\gamma(s))\,\gamma'(s)\,ds,$$

where $\gamma(s)$ $(0 \leq s \leq L)$ is a unit-speed parametrisation of ∂K. For an obstacle K in a uniform stream with circulation $2\pi c$, this formula gives, after some calculation and an application of the Flow Mapping Theorem,

$$F = \overline{\tfrac{1}{2}i\rho\int_{\partial K} (\overline{q}(z))^2\,dz} = -2\pi\rho c i.$$

This integral formula for F is Blasius' Theorem, and the formula $F = -2\pi\rho c i$ is the Kutta–Joukowski Theorem, which states that the force on such an obstacle K in a uniform stream acts in the vertical direction, and its magnitude depends only on the circulation $2\pi c$. This result suggests that aerofoils should be designed to give negative circulation.

One physically plausible requirement for the flow around an aerofoil is that the velocity function q is bounded in the flow region, and $q(z)$ tends to a limit as z tends to the trailing edge of the aerofoil; this is the Kutta–Joukowski Hypothesis. At the end of the solution to Exercise 5.2 it was pointed out that this hypothesis holds for a symmetric aerofoil with angle of attack ϕ if and only if $c = -2(a+b)\sin\phi$, in which case the circulation is indeed negative.

As you may imagine, the model of fluid flow described in this unit is rather simple, and the assumptions made in Section 1 do not hold exactly for any real fluid. For example, all fluids have some viscosity, essentially a 'stickiness' due to molecular interactions, which causes flows near solid boundaries to fail to satisfy the locally circulation-free condition. Also, there are some clear inconsistencies in the model's conclusions.

Nevertheless, the model does give reasonably accurate predictions in some circumstances, for example when the angle of attack of an aerofoil is quite small and certain parts of the flow are considered. The photograph in Figure 5.6 indicates that the actual flow around a slightly inclined aerofoil closely resembles the model of the flow obtained using a Joukowski function, at least away from the upper edge of the aerofoil, where the flow appears to separate from the aerofoil.

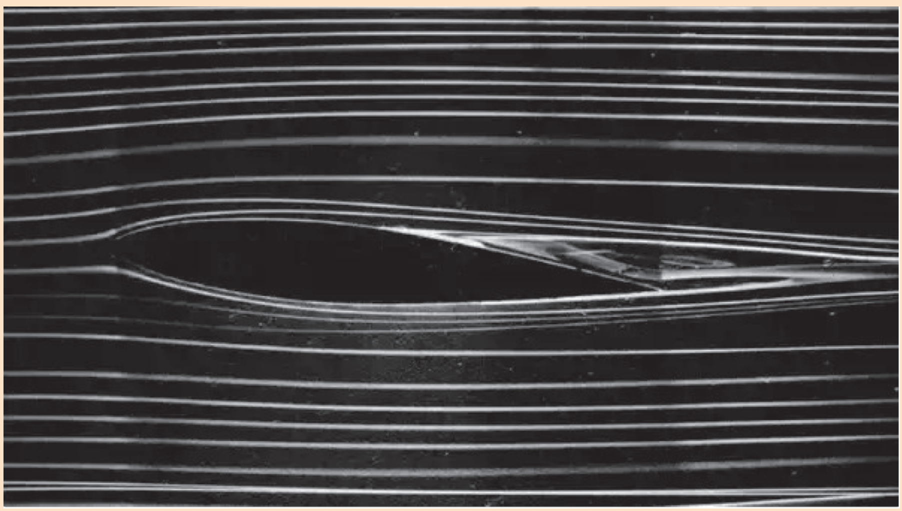


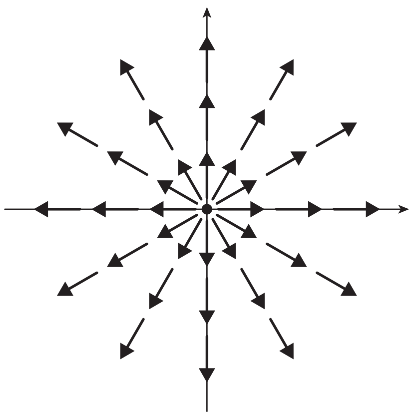
Figure 5.6 Boundary-layer separation on an inclined aerofoil

Finally, there are methods for improving the results obtained with the basic fluid flow model in this unit. For example, the Joukowski function can be replaced by a similar but more complicated conformal mapping called the *Kármán–Trefftz transformation*, which gives rise to a more realistic aerofoil whose trailing edge has a small positive acute angle, as in Figure 5.6, rather than a cusp.

Solutions to exercises

Solution to Exercise 1.1

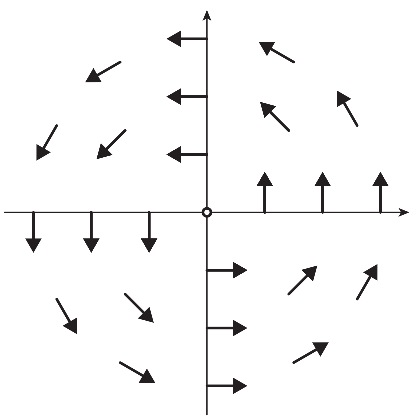
(a) For $q(z) = z$, all of the arrows for $z \neq 0$ have direction $\operatorname{Arg} z$, so they point radially outwards. At the origin, $q(z) = 0$, so there is a stagnation point there.



(b) By using properties of arguments from Subsection 2.3 of Unit A1 (for example, that an argument of z_1 minus an argument of z_2 is an argument of z_1/z_2), we have that the direction of $q(z) = i/\overline{z}$, for $z \neq 0$, is given by

$$\begin{aligned}\operatorname{Arg}(i/\overline{z}) &= \operatorname{Arg} i - \operatorname{Arg} \overline{z} + 2n\pi\\ &= \operatorname{Arg} i + \operatorname{Arg} z + 2n\pi\\ &= \operatorname{Arg} z + \pi/2 + 2n\pi,\end{aligned}$$

for some $n \in \mathbb{Z}$.

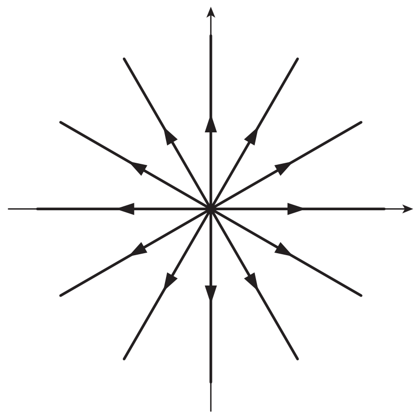


We draw a hollow dot at the origin to show that this point is excluded from the flow region $\mathbb{C} - \{0\}$.

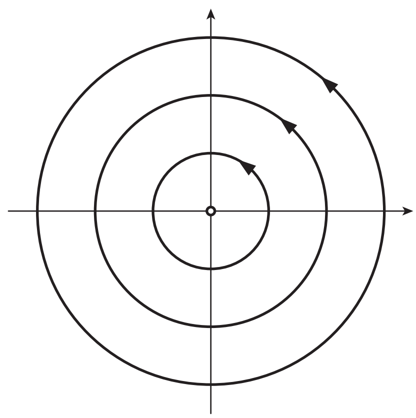
Solution to Exercise 1.2

In each case the streamlines can be sketched from the corresponding arrow diagram in the solution to Exercise 1.1, as shown in the following figures.

(a) The streamlines are rays from 0 with the flow outwards.



(b) The streamlines are circles around 0 with the flow anticlockwise.



Solution to Exercise 1.3

From equation (1.1), with $q(z) = 5e^{-i\pi/6}$ and $\theta = 2\pi/3$, we have

$$\begin{aligned}q_{2\pi/3}(z) &= \operatorname{Re}(\overline{5e^{-i\pi/6}}e^{2i\pi/3})\\ &= \operatorname{Re}(5e^{i\pi/6}e^{2i\pi/3})\\ &= \operatorname{Re}(5e^{5i\pi/6})\\ &= 5\cos\frac{5\pi}{6}\\ &= -\tfrac{5}{2}\sqrt{3}.\end{aligned}$$

Solution to Exercise 1.4

From equation (1.1) we have

$$q_\theta(z) = \operatorname{Re}(\overline{q(z)}e^{i\theta}).$$

With $\theta - \pi/2$ in place of θ, this becomes

$$\begin{aligned} q_{(\theta-\pi/2)}(z) &= \operatorname{Re}(\overline{q(z)}e^{i(\theta-\pi/2)}) \\ &= \operatorname{Re}(\overline{q(z)}e^{i\theta}e^{-i\pi/2}) \\ &= \operatorname{Re}(-i\overline{q(z)}e^{i\theta}). \end{aligned}$$

Now, if $w = u + iv$ is any complex number, then

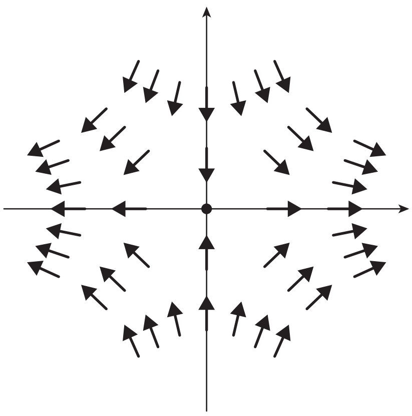
$$\operatorname{Re}(-iw) = \operatorname{Re}(v - iu) = v = \operatorname{Im} w.$$

It follows that

$$q_{(\theta-\pi/2)}(z) = \operatorname{Im}(\overline{q(z)}e^{i\theta}).$$

Solution to Exercise 1.5

(a) If $\overline{q}(z) = z$ is the conjugate velocity function, then the velocity function itself is $q(z) = \overline{z}$. To draw the arrow diagram for this velocity function note that $\overline{z}$ has the same real part as z, but an imaginary part of opposite sign. This gives a sketch like that below, using arrows of fixed length (except at 0, which is a stagnation point of the flow).



(b) If $\overline{q}(z) = 1/z$, then we have

$$q(z) = 1/\overline{z} \quad (z \in \mathbb{C} - \{0\}).$$

Now

$$\operatorname{Arg} q(z) = \operatorname{Arg}(1/\overline{z}) = \operatorname{Arg} z,$$

so the direction of $q(z)$ is radially outwards.

This leads to the same fixed-length arrow diagram as in the solution to Exercise 1.1(a), except that the origin should in this case be omitted because it is excluded from the domain of q.

(c) If $\overline{q}(z) = i/z$, then we have

$$q(z) = -i/\overline{z} \quad (z \in \mathbb{C} - \{0\}).$$

The diagram is like that of the solution to Exercise 1.1(b) but with all the arrows reversed.

Solution to Exercise 1.6

(a) The velocity function $q(z) = z$ $(z \in \mathbb{C})$ has conjugate function $\overline{q}(z) = \overline{z}$. This is not an analytic function (see Exercise 1.8 of Unit A4), so q is not a velocity function for an ideal flow, by Theorem 1.3.

(b) For $q(z) = i/\overline{z}$ $(z \in \mathbb{C} - \{0\})$, the conjugate velocity function is $\overline{q}(z) = -i/z$. This function is analytic on $\mathbb{C} - \{0\}$, so q is the velocity function of an ideal flow, by Theorem 1.3.

Solution to Exercise 1.7

(a) The flux of $q(z) = 1/\overline{z}$ across the unit circle Γ is, by Theorem 1.2,

$$\begin{aligned} \mathcal{F}_\Gamma &= \operatorname{Im} \int_\Gamma \overline{q}(z)\,dz \\ &= \operatorname{Im} \int_\Gamma \frac{1}{z}\,dz \\ &= \operatorname{Im}(2\pi i) \quad \text{(by Cauchy's Integral Formula)} \\ &= 2\pi. \end{aligned}$$

(b) Similarly, the circulation of $q(z) = -i/\overline{z}$ along Γ is

$$\begin{aligned} \mathcal{C}_\Gamma &= \operatorname{Re} \int_\Gamma \overline{q}(z)\,dz \\ &= \operatorname{Re} \int_\Gamma \frac{i}{z}\,dz \\ &= \operatorname{Re}(i \times 2\pi i) \\ &= -2\pi. \end{aligned}$$

(c) Since the conjugates of the velocity functions in parts (a) and (b) are both analytic, it follows from Theorem 1.3 that both flows are ideal flows. Therefore $\mathcal{C}_\Gamma = \mathcal{F}_\Gamma = 0$ for each simple-closed contour Γ in the domain $\mathcal{R} = \mathbb{C} - \{0\}$ whose inside also lies in $\mathcal{R}$. This does not contradict the answers to parts (a) and (b), because the inside of the unit circle Γ contains the origin, so it does not lie in $\mathbb{C} - \{0\}$.

Solution to Exercise 1.8

(a) By the Cauchy–Riemann Theorem and the Cauchy–Riemann Converse Theorem (Subsection 2.1 of Unit A4), if the partial derivatives of the real functions u and v with respect to x and y exist, and are continuous on $\mathcal{R}$, then $u + iv$ is an analytic function on $\mathcal{R}$ if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{on } \mathcal{R}.$$

Now, the velocity function $q = q_1 + iq_2$ has conjugate function $\overline{q} = q_1 - iq_2$, and q is a velocity function for an ideal flow on $\mathcal{R}$ if and only if $\overline{q}$ is analytic on $\mathcal{R}$ (by Theorem 1.3). Putting these results together, with $u = q_1$ and $v = -q_2$, we have that q is a velocity function for an ideal flow on $\mathcal{R}$ if and only if

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0 \quad \text{and} \quad \frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0 \quad \text{on } \mathcal{R}.$$

(b) The real and imaginary parts of

$$q(z) = \frac{i}{\overline{z}} = \frac{iz}{|z|^2} = \frac{i(x+iy)}{x^2+y^2},$$

are, respectively,

$$q_1(x,y) = -\frac{y}{x^2+y^2}, \quad q_2(x,y) = \frac{x}{x^2+y^2}.$$

The partial derivatives of q_1 and q_2 are

$$\frac{\partial q_1}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial q_1}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2},$$
$$\frac{\partial q_2}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial q_2}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}.$$

Hence we find, if $z \neq 0$, that

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0 \quad \text{and} \quad \frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0.$$

Therefore, by the result of part (a), $q(z) = i/\overline{z}$ ($z \in \mathbb{C} - \{0\}$) is a velocity function for an ideal flow.

Solution to Exercise 1.9

(a) For the velocity function $q(z) = z$ we have $q_1(x,y) = x$ and $q_2(x,y) = y$, so

$$\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0.$$

Hence, by Theorem 1.4(a), q is locally circulation-free. However, since

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 1 + 1 = 2 \neq 0,$$

the function q is not locally flux-free.

(b) For the velocity function $q(z) = iz$ we have $q_1(x,y) = -y$ and $q_2(x,y) = x$, so

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0.$$

Hence, by Theorem 1.4(b), q is locally flux-free. However, since

$$\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 1 - (-1) = 2 \neq 0,$$

the function q is not locally circulation-free.

Solution to Exercise 1.10

(a) From equation (1.1), with $q(z) = 3e^{7i\pi/12}$ and $\theta = -2\pi/3$, we have

$$\begin{aligned} q_{-2\pi/3}(z) &= \operatorname{Re}(3e^{-7i\pi/12}e^{-2i\pi/3}) \\ &= \operatorname{Re}(3e^{-5i\pi/4}) \\ &= 3\cos\left(-\frac{5\pi}{4}\right) \\ &= -3/\sqrt{2}. \end{aligned}$$

(b) The second direction is specified by

$$-ie^{-2i\pi/3} = e^{-i\pi/2}e^{-2i\pi/3} = e^{-7i\pi/6} = e^{5i\pi/6},$$

so the required component is

$$\begin{aligned} q_{5\pi/6}(z) &= \operatorname{Re}(3e^{-7i\pi/12}e^{5i\pi/6}) \\ &= \operatorname{Re}(3e^{i\pi/4}) \\ &= 3\cos\frac{\pi}{4} \\ &= 3/\sqrt{2}. \end{aligned}$$

Solution to Exercise 1.11

We have

$$\begin{aligned} \overline{q}(z)e^{i\theta} &= \operatorname{Re}(\overline{q}(z)e^{i\theta}) + i\operatorname{Im}(\overline{q}(z)e^{i\theta}) \\ &= q_\theta(z) + iq_{(\theta-\pi/2)}(z), \end{aligned}$$

by equations (1.2). Taking the complex conjugate of both sides, we obtain

$$q(z)e^{-i\theta} = q_\theta(z) - iq_{(\theta-\pi/2)}(z),$$

so

$$q(z) = (q_\theta(z) - iq_{(\theta-\pi/2)}(z))e^{i\theta}.$$

Solution to Exercise 1.12

(a) The conjugate velocity function is

$$\overline{q}(z) = 1 - \frac{1}{z^2},$$

so, by Theorem 1.2, we obtain

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \left(1 - \frac{1}{z^2}\right) dz.$$

This integral can be shown to be zero either by using the standard parametrisation for the contour Γ, or by applying the Residue Theorem. Hence we have

$$\mathcal{C}_\Gamma = \mathcal{F}_\Gamma = 0.$$

(b) **(i)** The conjugate velocity function $\overline{q}$ is analytic on $\mathbb{C} - \{0\}$. Hence, by Theorem 1.3, q is a velocity function for an ideal flow on this region.

(ii) The real and imaginary parts of

$$q(z) = 1 - \frac{1}{\overline{z}^2} = 1 - \frac{z^2}{|z|^4} = 1 - \frac{(x+iy)^2}{(x^2+y^2)^2}$$

are, respectively,

$$q_1(x,y) = 1 - \frac{x^2 - y^2}{(x^2+y^2)^2}$$

and

$$q_2(x,y) = -\frac{2xy}{(x^2+y^2)^2}.$$

The partial derivatives of q_1 and q_2 are

$$\frac{\partial q_1}{\partial x} = \frac{2x(x^2-3y^2)}{(x^2+y^2)^3}, \quad \frac{\partial q_1}{\partial y} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3},$$
$$\frac{\partial q_2}{\partial x} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}, \quad \frac{\partial q_2}{\partial y} = \frac{2x(3y^2-x^2)}{(x^2+y^2)^3}.$$

Hence we find, if $z \neq 0$, that

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0 \quad \text{and} \quad \frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0.$$

By Theorem 1.4, q is locally both flux-free and circulation-free. Hence q is a velocity function for an ideal flow.

Solution to Exercise 2.1

The complex potential Ω satisfies

$$\Omega'(z) = \overline{q}(z).$$

With $\Omega(z) = \Phi(z) + i\Psi(z)$ we have

$$\begin{aligned}\Omega'(x+iy) &= \frac{\partial \Phi}{\partial x}(x,y) + i\frac{\partial \Psi}{\partial x}(x,y)\\ &= \frac{\partial \Psi}{\partial y}(x,y) + i\frac{\partial \Psi}{\partial x}(x,y),\end{aligned}$$

by the Cauchy–Riemann Theorem and its converse.

On taking complex conjugates we obtain

$$q(x+iy) = \overline{\Omega'(z)} = \frac{\partial \Psi}{\partial y}(x,y) - i\frac{\partial \Psi}{\partial x}(x,y).$$

Solution to Exercise 2.2

The conjugate of $q(z)$ is $\overline{q}(z) = iz$, so a complex potential function for the flow is

$$\Omega(z) = iz^2/2.$$

Writing $z = x + iy$, we obtain

$$\tfrac{1}{2}iz^2 = \tfrac{1}{2}i(x+iy)^2 = -xy + \tfrac{1}{2}(x^2-y^2)i,$$

so $\operatorname{Im}\Omega(z) = \frac{1}{2}(x^2 - y^2)$.

Hence, by Theorem 2.1, the streamlines have equations of the form

$$x^2 - y^2 = k, \quad \text{where } k \text{ is a constant.}$$

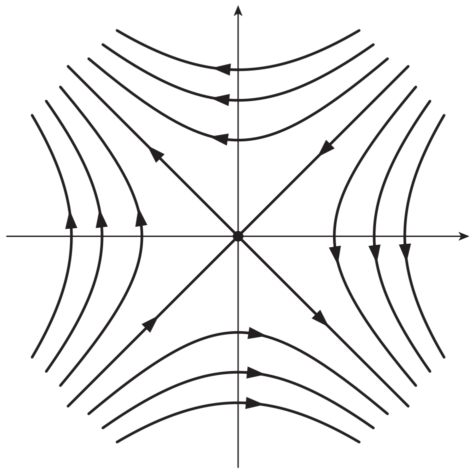
For $k \neq 0$ these streamlines are branches of hyperbolas obtained by rotating the curves $xy = k$ shown in Figure 2.4 through an angle of $-\pi/4$ about 0, and for $k = 0$ the streamlines form the parts of the lines $y = \pm x$ in the four quadrants together with a degenerate streamline at the origin.

We can use the formula for the velocity function,

$$q(z) = -i\overline{z} = -i(x - iy) = -y - ix,$$

to establish the direction of flow along the streamlines. For example, on $y = x$ in the upper-right quadrant, $q(z)$ points inwards along this line.

Hence, by using the continuity of q, the branches of the hyperbolas that cross the positive x-axis must have arrows indicating that the flow is downwards along these streamlines. You can use a similar approach to determine the arrow directions on the other branches of these hyperbolas.



Solution to Exercise 2.3

A complex potential Ω is given by $\Omega'(z) = \overline{q}(z)$. Here we have

$$q(z) = \frac{-1+8i}{\overline{z}} \quad (z \in \mathbb{C} - \{0\}),$$

so we require for Ω a primitive of

$$\overline{q}(z) = -\frac{1+8i}{z}.$$

We need to choose a simply connected region in $\mathbb{C} - \{0\}$ to be the domain of Ω. If we choose the cut plane $\mathbb{C}_\pi$, then we can take

$$\begin{aligned}\Omega(z) &= -(1+8i)\operatorname{Log} z \\ &= -(1+8i)(\log|z| + i\operatorname{Arg} z).\end{aligned}$$

The stream function is then

$$\operatorname{Im}\Omega(z) = -(8\log|z| + \operatorname{Arg} z).$$

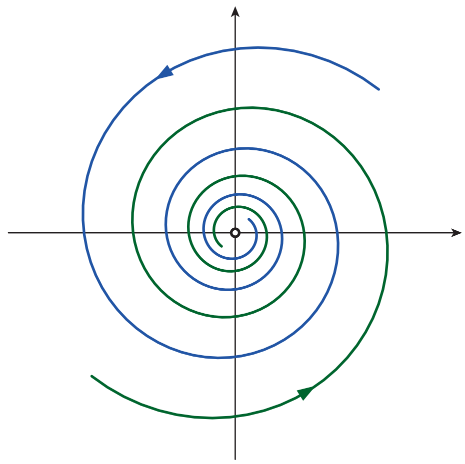
The equation for streamlines is $\operatorname{Im}\Omega(z) = k$, where k is a constant; that is,

$$-8\log|z| = k + \operatorname{Arg} z.$$

Further manipulation of this equation gives

$$|z| = e^{-k/8}e^{-(\operatorname{Arg} z)/8} = Ke^{-(\operatorname{Arg} z)/8},$$

where $K = e^{-k/8}$ is a (positive) constant. As $\operatorname{Arg} z$ increases from $-\pi$ to π, $|z|$ decreases so each streamline includes part of a spiral directed inwards. By using complex potential functions defined with different logarithm functions $\operatorname{Log}_\phi(z) = \log|z| + i\operatorname{Arg}_\phi(z)$ for values of ϕ greater than π, we find that these streamlines form spirals that approach closer and closer to 0, as shown in the following figure.



The sink strength and vortex strength are the (absolute values of) the flux across and circulation along any simple-closed contour Γ that surrounds the origin, respectively. These are, in turn, the imaginary and real parts of

$$\int_\Gamma \overline{q}(z)\,dz = \int_\Gamma -\frac{1+8i}{z}\,dz,$$

which, by Cauchy's Integral Formula, has the value

$$-2\pi i(1+8i) = 16\pi - 2\pi i.$$

Hence the flow has sink strength $|-2\pi| = 2\pi$ and vortex strength 16π.

Remark: This flow is a combination of a sink and a vortex, called a *whirlpool* or *spiral vortex*.

Solution to Exercise 2.4

(a) We have to show that

$$\overline{\Omega'(z)} = q(z), \quad \text{for } z \in \mathcal{R},$$

where $\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \le h\}$. On $\mathcal{R}$,

$$\begin{aligned}\Omega'(z) &= \operatorname{Log}' z - \operatorname{Log}'(z-h) \\ &= \frac{1}{z} - \frac{1}{z-h},\end{aligned}$$

so

$$\overline{\Omega'(z)} = \frac{1}{\overline{z}} - \frac{1}{\overline{z}-h} = q(z),$$

as required.

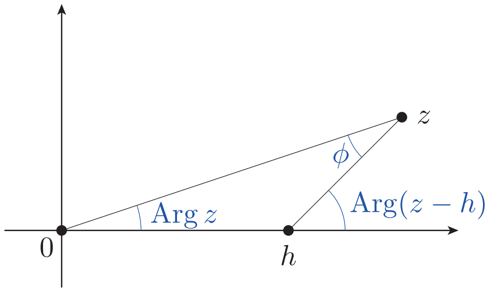
The complex potential

$$\Omega(z) = \operatorname{Log} z - \operatorname{Log}(z-h)$$

leads to the stream function

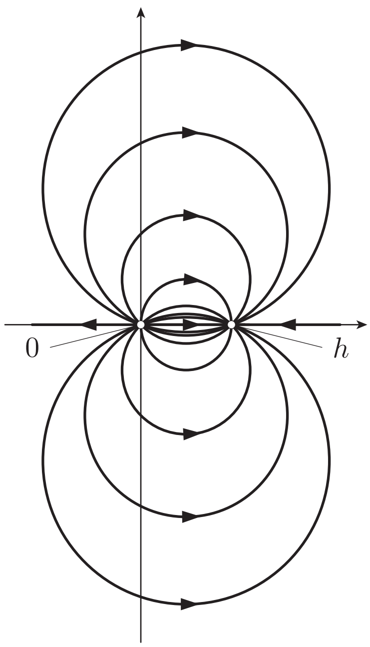
$$\Psi(z) = \operatorname{Im}\Omega(z) = \operatorname{Arg} z - \operatorname{Arg}(z-h).$$

(b) The streamlines are the paths given by $\Psi(z) = k$. Suppose first that z is in the upper half-plane.



The angle ϕ marked on the figure is equal to $\operatorname{Arg}(z-h) - \operatorname{Arg} z$, since the exterior angle of any triangle is equal to the sum of its two opposite interior angles. Hence as z moves along a streamline, the angle ϕ remains constant. It follows, by the geometry hint, that z moves along a circular arc passing through 0 and h. By symmetry, the centre of the circle lies on the line $x = \frac{1}{2}h$.

A similar conclusion is reached by considering z to lie below the real axis. The real axis itself contains three streamlines, namely $(-\infty, 0)$, $(0, h)$ and (h, ∞). The flow directions on the streamlines follow from the facts that $z = 0$ is a source and $z = h$ is a sink. Therefore the streamline diagram is as follows.



Solution to Exercise 2.5

Taking the limit, we have

$$\begin{aligned}\lim_{h\to 0} q_h(z) &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\overline{z}} - \frac{1}{\overline{z}-h}\right)\\ &= \lim_{h\to 0} \frac{-h}{h\overline{z}(\overline{z}-h)}\\ &= -\frac{1}{\overline{z}^2},\end{aligned}$$

which is the velocity function for a doublet. (The streamlines near a doublet are shown in Figure 2.7.)

Solution to Exercise 2.6

(a) The velocity function is

$$q(z) = 2\left(1 + \frac{1}{\overline{z}}\right).$$

A complex potential for this flow is

$$\Omega(z) = 2(z + \operatorname{Log} z) \quad (z \in \mathbb{C}_\pi),$$

and the corresponding stream function is

$$\operatorname{Im}\Omega(z) = 2(\operatorname{Im} z + \operatorname{Arg} z) \quad (z \in \mathbb{C}_\pi).$$

(b) The only stagnation point is the unique solution of the equation $q(z) = 0$, which is $z = -1$.

(c) By part (a) the streamlines have equations of the form

$$y + \operatorname{Arg} z = k,$$

where $z = x + iy$ and k is a constant. We have

$$\lim_{z\to -1} \operatorname{Im} z = 0 \quad \text{and} \quad \lim_{z\to -1} \operatorname{Arg} z = \pm\pi,$$

where the plus or minus sign is chosen depending on whether the limit is taken from above or below the real axis, respectively. Therefore any streamlines that approach the stagnation point from the upper half-plane or lower half-plane must have equation

$$y + \operatorname{Arg} z = \pi \quad \text{or} \quad y + \operatorname{Arg} z = -\pi.$$

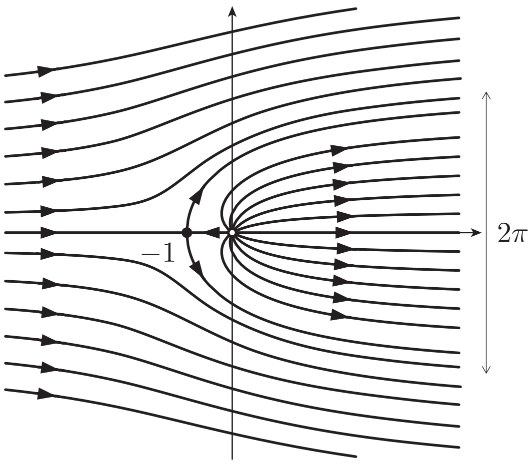
The streamline with equation $y + \operatorname{Arg} z = \pi$ lies in the horizontal strip $\{x + iy : 0 < y < \pi\}$, meets each horizontal line $y = c$, $0 < c < \pi$, exactly once, and crosses the imaginary axis at $i\pi/2$.

Similarly, there is a streamline with equation $y + \operatorname{Arg} z = -\pi$ that approaches -1 and lies in the horizontal strip $\{x + iy : -\pi < y < 0\}$.

(d) The streamline with equation $y + \operatorname{Arg} z = \pi$ has the property that as y increases on the curve towards the value π, the quantity $\operatorname{Arg}(x + iy)$ decreases to 0.

Similarly, the streamline with equation $y + \operatorname{Arg} z = -\pi$ has the property that as y decreases on the curve towards the value $-\pi$, the quantity $\operatorname{Arg}(x + iy)$ increases to 0.

Knowing the approximate shape of these two streamlines, together with the facts that for large z the flow is approximately uniform and that there is a source at 0, enables us to produce a sketch of the streamlines, as below.



Remark: The two streamlines emerging from the stagnation point -1 can be considered to be the boundary of a large blunt object, of width approaching 2π, placed in a uniform flow, as shown in the figure.

Solution to Exercise 2.7

If q is the velocity function for an ideal flow with flow region $\mathbb{C}$, and it is bounded, then there exists a real number K such that

$$|q(z)| \leq K, \quad \text{for } z \in \mathbb{C}.$$

But $\overline{q}$ is an analytic function with the same domain as q, and it is therefore entire. Also, we have

$$|\overline{q}(z)| = |q(z)|, \quad \text{for } z \in \mathbb{C},$$

so $\overline{q}$ is also bounded. It follows from Liouville's Theorem that $\overline{q}$ is a constant function, and hence that q is constant also. The corresponding flow is therefore uniform.

Solution to Exercise 3.1

(a) The function J is one-to-one on the region $\{z : |z| > 1\}$, and for $x \in (0, \infty)$ we have

$$J(x) = x + \frac{1}{x},$$

so the restriction of J to the interval $[1, \infty)$ is a one-to-one real function. It follows that J is monotonic on this interval (in fact it is increasing since $J'(x) = 1 - 1/x^2 > 0$ for $x \in (1, \infty)$), so J maps $[1, r]$ onto $[J(1), J(r)] = [2, r + 1/r]$.

(b) For $y \in (0, \infty)$ we have

$$J(iy) = iy + \frac{1}{iy} = i\left(y - \frac{1}{y}\right) = i\phi(y),$$

where $\phi(y) = y - 1/y$. So the restriction of J to $\{iy : y \geq 1\}$ is a one-to-one function which takes values on the positive imaginary axis, for $y > 0$. Hence J is a one-to-one mapping of the line segment from i to ir onto the line segment from $J(i) = 0$ to $J(ir) = i(r - 1/r)$.

Solution to Exercise 3.2

(a) The given circle is

$$C_r = \{z : |z| = r\} = \{re^{it} : 0 \leq t \leq 2\pi\}.$$

The image under J_a of a typical point in C_r is

$$\begin{aligned} J(re^{it}) &= re^{it} + \frac{a^2}{r}e^{-it} \\ &= \left(r + \frac{a^2}{r}\right)\cos t + i\left(r - \frac{a^2}{r}\right)\sin t. \end{aligned}$$

Thus the parametric equations of the image are

$$u = \left(r + \frac{a^2}{r}\right)\cos t, \quad v = \left(r - \frac{a^2}{r}\right)\sin t,$$

for $0 \leq t \leq 2\pi$.

Now $\cos^2 t + \sin^2 t = 1$, so the image of the circle C_r is the ellipse with equation

$$\frac{u^2}{(r + a^2/r)^2} + \frac{v^2}{(r - a^2/r)^2} = 1.$$

Also, if $z = re^{it}$ traverses the circle C_r once, then the image point $w = f(re^{it})$ traverses the ellipse once, so J_a is a one-to-one mapping of C_r onto the ellipse.

(b) The image under J_a of a typical point on the ray is

$$\begin{aligned} J_a(te^{i\theta}) &= te^{i\theta} + \frac{a^2}{t}e^{-i\theta} \\ &= \left(t + \frac{a^2}{t}\right)\cos\theta + i\left(t - \frac{a^2}{t}\right)\sin\theta. \end{aligned}$$

Thus the parametric equations of the image are

$$u = \left(t + \frac{a^2}{t}\right)\cos\theta, \quad v = \left(t - \frac{a^2}{t}\right)\sin\theta,$$

for $a < t < \infty$.

Now we can use the identity

$$\left(t + \frac{a^2}{t}\right)^2 - \left(t - \frac{a^2}{t}\right)^2 = 4a^2$$

to eliminate the parameter t, so the image of the ray has equation

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4a^2,$$

which is a hyperbola. Since $\cos\theta > 0$ and $\sin\theta > 0$ for $0 < \theta < \pi/2$, the image of the ray is the part of this hyperbola lying in the upper-right quadrant of the w-plane. As the ray is traversed once, its image under J_a is traversed once, so J_a is a one-to-one mapping of the ray onto the part of the hyperbola that lies in the upper-right quadrant.

Solution to Exercise 3.3

The ellipse E is of the same form as that considered in Exercise 3.2(a), but rotated about 0 through an angle $\pi/2$. By equation (3.1) we have the decomposition

$$J_\alpha = R_{-\pi/2}^{-1} \circ J_a \circ R_{-\pi/2},$$

where $\alpha = ia = ae^{i\pi/2}$. Using this decomposition of J_{ia}, we find that $R_{-\pi/2}$ maps the circle C_r onto itself, then J_a maps this circle onto the ellipse in Exercise 3.2(a), and finally $R_{-\pi/2}^{-1} = R_{\pi/2}$ maps this ellipse onto E. Hence J_{ia} maps C_r onto E.

Since J_{ia} is a one-to-one conformal mapping from $\{z : |z| > a\}$ onto $\mathbb{C} - L(-2ia, 2ia)$, by Theorem 3.3(b), and E surrounds $L(-2ia, 2ia)$, it follows that J_{ia} is a one-to-one conformal mapping from $\{z : |z| > r\}$ onto the unbounded region with E as its boundary.

Solution to Exercise 3.4

By Theorem 3.1, the Joukowski function J is a one-to-one conformal mapping from $\mathbb{C} - \{z : |z| \leq 1\}$ onto $\mathbb{C} - [-2, 2]$. Also, by Exercise 3.1(a), with $r = 2$, we have

$$J([1, 2]) = \left[2, \tfrac{5}{2}\right],$$

and, by similar reasoning (or using the fact that J is an odd function, that is, $J(-z) = -J(z)$ for $z \in \mathbb{C} - \{0\}$), we have

$$J([-2, -1]) = \left[-\tfrac{5}{2}, -2\right].$$

Hence J maps $\mathbb{C} - K$ onto the region that is the complement of

$$\left[-\tfrac{5}{2}, -2\right] \cup [-2, 2] \cup \left[2, \tfrac{5}{2}\right] = \left[-\tfrac{5}{2}, \tfrac{5}{2}\right],$$

as required.

Solution to Exercise 3.5

In the proof of Theorem 3.1 you saw that the Joukowski function can be expressed as the composition of three simpler functions:

$$z_1 = \frac{z+1}{z-1}, \quad z_2 = z_1^2, \quad w = \frac{2z_2 + 2}{z_2 - 1},$$

applied in this order.

We suppose that the acute angle between the circle C and the unit circle C_1 at both points ± 1 is θ, as shown in the figure in the exercise. Then the Möbius transformation

$$z_1 = \frac{z+1}{z-1}$$

maps C to an extended line through 0, making an angle of θ with the imaginary axis and lying in the upper-left quadrant and the lower-right quadrant.

Next, $z_2 = z_1^2$ maps this extended line through 0 to an extended half-line with endpoint at 0, making an angle of 2θ with the negative real axis and lying in the lower half-plane.

Finally, the Möbius transformation

$$w = \frac{2z_2 + 2}{z_2 - 1}$$

maps this extended half-line with endpoint at 0 to an arc of a circle from 2 to -2, making an angle of 2θ with the line segment $[-2, 2]$ at each of its endpoints, and lying in the upper half-plane.

Remark: Note that the critical points of J are ± 1 and at these points the angle θ is doubled under J. Also note that under J, the arcs of C that lie inside C_1 and outside C_1 are both mapped to the *same* arc from 2 to -2.

Solution to Exercise 4.1

(a) The condition for stagnation points is $q(z) = 0$, or equivalently $\overline{q}(z) = 0$. Here we have

$$\overline{q}(z) = 1 - \frac{a^2}{z^2}, \quad \text{for } z \in \mathbb{C} - \{0\},$$

so the stagnation points satisfy

$$z^2 - a^2 = (z+a)(z-a) = 0.$$

Hence the only stagnation points are $z = \pm a$.

(b) Substitution of $z = ae^{it}$ into $q(z) = 1 - a^2/\overline{z}^2$ gives

$$\begin{aligned} q(ae^{it}) &= 1 - \frac{a^2}{a^2 e^{-2it}} \\ &= 1 - e^{2it} \\ &= (e^{-it} - e^{it})e^{it} \\ &= (-2i \sin t)e^{it}, \quad \text{for } 0 \le t \le 2\pi. \end{aligned}$$

This gives the velocity of the flow at $z = ae^{it}$ on the circle $\{z : |z| = a\}$. The flow speed here is

$$\left|q(ae^{it})\right| = 2|\sin t|.$$

Solution to Exercise 4.2

The stagnation points of the flow satisfy the equation $q(z) = 0$, which is equivalent to the quadratic equation

$$z^2 - icz - a^2 = 0.$$

Hence the stagnation points are

$$z = \tfrac{1}{2}(ic \pm \sqrt{4a^2 - c^2}).$$

(a) For $-2a < c < 0$ we have $4a^2 - c^2 > 0$, so the stagnation points have real parts $\pm\frac{1}{2}\sqrt{4a^2 - c^2}$ and imaginary part $\frac{1}{2}c$. The modulus in each case is

$$|z| = \tfrac{1}{2}\sqrt{(4a^2 - c^2) + c^2} = a,$$

so both stagnation points lie on the circle $\{z : |z| = a\}$. They are below the real axis (since $c < 0$) and symmetrically placed on either side of the imaginary axis.

(b) For $c = -2a$ we have $4a^2 - c^2 = 0$, so $z = \frac{1}{2}ic = -ia$, a single stagnation point.

(c) For $c < -2a$ we have $4a^2 - c^2 < 0$, so the stagnation points are z_+ and z_-, where

$$z_\pm = \tfrac{1}{2}i(c \pm \sqrt{c^2 - 4a^2}).$$

Both of these points are on the imaginary axis and below the real axis (since $c < 0$ and $|c| > \sqrt{c^2 - 4a^2}$). Since $c < -2a$ we have

$$\operatorname{Im} z_- = \tfrac{1}{2}(c - \sqrt{c^2 - 4a^2}) < \tfrac{1}{2}c < -a,$$

so $|z_-| > a$. Hence z_- lies on the negative imaginary axis below the circle $\{z : |z| = a\}$.

On the other hand, z_+ lies inside the circle $\{z : |z| = a\}$ because $z_+ z_- = -a^2$, so $|z_+| = a^2/|z_-| < a$.

Solution to Exercise 4.3

We show that $q(z) = 1$ satisfies the three properties given in the statement of the Obstacle Problem.

(a) Since $q(z) = 1$, $\lim_{z\to\infty} q(z) = 1$.

(b) A complex potential function for q is

$$\Omega(z) = z \quad (z \in \mathbb{C} - K).$$

Let $K = L(a + ik, b + ik)$, where $a, b, k \in \mathbb{R}$ and $a \neq b$. If $\alpha \in \partial K = K$, then

$$\lim_{z\to\alpha} \operatorname{Im} \Omega(z) = \lim_{z\to\alpha} \operatorname{Im} z = k.$$

(c) Since $\overline{q}$ has an analytic extension to the whole of $\mathbb{C}$, we have $\mathcal{C}_\Gamma = 0$ for any simple-closed contour Γ surrounding K.

Solution to Exercise 4.4

The translation $f(z) = z - \beta$ is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_a$. Also, f satisfies the Laurent series condition of the Flow Mapping Theorem with $a_0 = -\beta$ and $a_{-1} = a_{-2} = \cdots = 0$. Thus the velocity function that solves the Obstacle Problem for K with circulation $2\pi c$ around K is

$$\begin{aligned} q(z) &= q_{a,c}(f(z))\overline{f'(z)} \\ &= \overline{1 - \frac{a^2}{(z-\beta)^2} - \frac{ic}{z-\beta}}, \end{aligned}$$

since $\overline{f'(z)} = 1$.

Solution to Exercise 4.5

As in Example 4.1, we use the fact that $f(z) = J^{-1}(z)$ is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_1$, which satisfies the Laurent series condition of the Flow Mapping Theorem. Thus a suitable complex potential function is

$$\Omega(z) = \Omega_{a,c}(f(z)),$$

where $f(z) = J^{-1}(z)$ and $a = 1$.

Using the hint, we have

$$\begin{aligned}\Omega(z) &= \Omega_{1,c}(J^{-1}(z))\\ &= J(J^{-1}(z)) - ic\,\mathrm{Log}(J^{-1}(z))\\ &= z - ic\,\mathrm{Log}(J^{-1}(z)).\end{aligned}$$

The velocity function is $q(z) = \overline{\Omega'(z)}$. By differentiating $\Omega(z) = z - ic\,\mathrm{Log}(J^{-1}(z))$ and using Lemma 4.1(c), we obtain

$$\begin{aligned}\Omega'(z) &= 1 - \frac{ic(J^{-1})'(z)}{J^{-1}(z)}\\ &= 1 - \frac{ic}{J^{-1}(z)}\left(\frac{J^{-1}(z)}{z\sqrt{1-4/z^2}}\right).\end{aligned}$$

Hence

$$q(z) = 1 - \overline{\left(\frac{ic}{z\sqrt{1-4/z^2}}\right)},$$

as required.

Solution to Exercise 4.6

By Theorem 3.2, the function

$$f(z) = J_a^{-1}(z) = \tfrac{1}{2}\left(z + z\sqrt{1-4a^2/z^2}\right)$$

is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_a$.

Now we show that $f = J_a^{-1}$ satisfies the Laurent series condition of the Flow Mapping Theorem. We use the binomial series

$$\begin{aligned}&(1-4a^2/z^2)^{1/2}\\ &= 1 + \frac{1}{2}\left(-\frac{4a^2}{z^2}\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(-\frac{4a^2}{z^2}\right)^2 + \cdots\\ &= 1 - \frac{2a^2}{z^2} - \frac{2a^4}{z^4} - \cdots, \quad \text{for } |z| > 2a.\end{aligned}$$

Hence

$$\begin{aligned}J_a^{-1}(z) &= \frac{1}{2}\left(z + z\left(1 - \frac{2a^2}{z^2} - \frac{2a^4}{z^4} - \cdots\right)\right)\\ &= z - \frac{a^2}{z} - \frac{a^4}{z^3} - \cdots, \quad \text{for } |z| > 2a,\end{aligned}$$

as required.

It follows from the Flow Mapping Theorem that the required flow velocity function is

$$\begin{aligned}q(z) &= q_{a,c}(J_a^{-1}(z))\overline{(J_a^{-1})'(z)}\\ &= \overline{\left(1 - \frac{a^2}{(J_a^{-1}(z))^2} - \frac{ic}{J_a^{-1}(z)}\right)}\,\overline{(J_a^{-1})'(z)}.\end{aligned}$$

Using the identities

$$(J_a^{-1})'(z) = \left(1 - a^2/(J_a^{-1}(z))^2\right)^{-1}$$

and

$$J_a^{-1}(z) - a^2/J_a^{-1}(z) = z\sqrt{1-4a^2/z^2},$$

from Lemma 4.1 with $\alpha = a$, we obtain

$$\begin{aligned}q(z) &= \overline{\left(1 - \frac{a^2}{(J_a^{-1}(z))^2} - \frac{ic}{J_a^{-1}(z)}\right)}\\ &\quad\times \overline{\left(1 - a^2/(J_a^{-1}(z))^2\right)^{-1}}\\ &= 1 - \overline{\left(\frac{ic}{J_a^{-1}(z) - a^2/J_a^{-1}(z)}\right)}\\ &= 1 - \overline{\left(\frac{ic}{z\sqrt{1-4a^2/z^2}}\right)},\end{aligned}$$

as required.

Solution to Exercise 4.7

The function J_a^{-1} is a one-to-one conformal mapping from $\mathbb{C} - [-2a, 2a]$ onto $\mathbb{C} - K_a = \{w : |w| \leq a\}$. This function maps the elliptical boundary ∂K of the given obstacle to the circular boundary $\partial K_r = C_r$ of the closed disc $K_r = \{w : |w| \leq r\}$. Hence the restriction f of J_a^{-1} to $\mathbb{C} - K$ is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_r$.

Also, by Exercise 4.6, we know that the mapping J_a^{-1} satisfies the Laurent series condition of the Flow Mapping Theorem, and hence f does also.

It follows from the Flow Mapping Theorem and Lemma 4.1(c) that the flow velocity function is

$$
\begin{aligned}
q(z) &= q_{r,c}(J_a^{-1}(z))\overline{(J_a^{-1})'(z)} \\
&= \overline{\left(1 - \frac{r^2}{(J_a^{-1}(z))^2} - \frac{ic}{J_a^{-1}(z)}\right)}\,\overline{(J_a^{-1})'(z)} \\
&= \overline{\left(1 - \frac{r^2}{(J_a^{-1}(z))^2} - \frac{ic}{J_a^{-1}(z)}\right)} \\
&\quad \times \overline{\left(1 - a^2/(J_a^{-1}(z))^2\right)^{-1}} \\
&= \overline{\left(1 - \frac{a^2}{(J_a^{-1}(z))^2} - \frac{r^2-a^2}{(J_a^{-1}(z))^2} - \frac{ic}{J_a^{-1}(z)}\right)} \\
&\quad \times \overline{\left(1 - a^2/(J_a^{-1}(z))^2\right)^{-1}} \\
&= 1 - \overline{\left(\frac{r^2 - a^2 + icJ_a^{-1}(z)}{(J_a^{-1}(z))^2 - a^2}\right)},
\end{aligned}
$$

as required.

Solution to Exercise 4.8

The stream function is

$$\operatorname{Im}\Omega(z) = y\left(1 - \frac{a^2}{x^2+y^2}\right),$$

which is constant on each streamline. Thus the streamline through $2ia$ (at which point $x = 0$ and $y = 2a$) has the equation

$$y\left(1 - \frac{a^2}{x^2+y^2}\right) = 2a\left(1 - \frac{a^2}{4a^2}\right) = \tfrac{3}{2}a.$$

Observe that

$$\frac{1}{x^2+y^2} = \frac{1}{|z|^2} \to 0 \text{ as } z \to \infty,$$

so

$$y = \tfrac{3}{2}a \Big/ \left(1 - \frac{a^2}{x^2+y^2}\right) \to \tfrac{3}{2}a \text{ as } z \to \infty$$

along the streamline. Hence this streamline has $y = \frac{3}{2}a$ as a horizontal asymptote.

Solution to Exercise 5.1

(a) The function J_a^{-1} maps the boundary of the aerofoil ∂K onto the circle C, and it is a one-to-one conformal mapping from $\mathbb{C} - K$ onto the region that forms the outside of C. Furthermore, the translation $g(w) = w + b$ maps C onto ∂K_{a+b}, where $K_{a+b} = \{w : |w| \le a+b\}$. Thus the composite function

$$f(z) = J_a^{-1}(z) + b$$

maps ∂K onto ∂K_{a+b}, and it is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_{a+b}$.

(b) Since J_a^{-1} satisfies the Laurent series condition of the Flow Mapping Theorem, so does f.

By the Flow Mapping Theorem (with $a+b$ in place of a), a suitable complex potential for the flow is

$$
\begin{aligned}
\Omega(z) &= (\Omega_{a+b,c} \circ f)(z) \\
&= \Omega_{a+b,c}(J_a^{-1}(z) + b) \\
&= w + b + \frac{(a+b)^2}{w+b} - ic\operatorname{Log}(w+b),
\end{aligned}
$$

where $w = J_a^{-1}(z)$.

(c) By differentiating Ω or using the formula given in the Flow Mapping Theorem, we obtain

$$q(z) = \overline{\left(1 - \frac{(a+b)^2}{(w+b)^2} - \frac{ic}{w+b}\right)(J_a^{-1})'(z)},$$

where, by Lemma 4.1(c),

$$
\begin{aligned}
(J_a^{-1})'(z) &= \left(1 - \frac{a^2}{(J_a^{-1}(z))^2}\right)^{-1} \\
&= \left(1 - \frac{a^2}{w^2}\right)^{-1} \\
&= \frac{w^2}{w^2-a^2}.
\end{aligned}
$$

It follows that, expressed in terms of $w = J_a^{-1}(z)$, the velocity function is

$$q(z) = \overline{\frac{w^2}{w^2-a^2}\left(1 - \frac{(a+b)^2}{(w+b)^2} - \frac{ic}{w+b}\right)},$$

as required.

(d) To find the behaviour of $q(z)$ as $z \to 2a$, and hence as $w = J_a^{-1}(z) \to a$, we do some rearranging using the hint:

$$\begin{aligned}
&\frac{w^2}{w^2-a^2}\left(1 - \frac{(a+b)^2}{(w+b)^2} - \frac{ic}{w+b}\right) \\
&= \frac{w^2}{w^2-a^2}\left(\frac{(w+a+2b)(w-a)}{(w+b)^2} - \frac{ic}{w+b}\right) \\
&= \frac{w^2(w+a+2b)}{(w+a)(w+b)^2} - \frac{icw^2}{(w+b)(w^2-a^2)}.
\end{aligned}$$

Now,

$$\frac{w^2(w+a+2b)}{(w+a)(w+b)^2} \to \frac{a^2(2a+2b)}{(2a)(a+b)^2} = \frac{a}{a+b}$$

as $w \to a$. Also, if $c \neq 0$, then

$$\frac{icw^2}{(w+b)(w^2-a^2)} \to \infty \text{ as } w \to a,$$

so $\lim\limits_{z\to 2a} q(z)$ does not exist. However, if $c = 0$, then the limiting velocity as $z \to 2a$ is $a/(a+b)$.

Solution to Exercise 5.2

(a) Since $\alpha = ae^{-i\phi}$ and $a > 0$, we have

$$J_\alpha = R_\phi^{-1} \circ J_a \circ R_\phi.$$

If B is the circle $R_\phi^{-1}(C)$, then

$$C = R_\phi(B).$$

It follows that

$$\begin{aligned}
\partial K &= (R_\phi^{-1} \circ J_a)(C) \\
&= (R_\phi^{-1} \circ J_a \circ R_\phi)(B) \\
&= J_\alpha(B).
\end{aligned}$$

Let D be the closed disc with boundary B which has centre $-be^{-i\phi}$ and radius $a+b$. The function J_α^{-1} maps $\mathbb{C} - K$ onto $\mathbb{C} - D$, and the translation $g(w) = w + be^{-i\phi}$ maps $\mathbb{C} - D$ onto $\mathbb{C} - K_{a+b}$.

Thus the function

$$f(z) = J_\alpha^{-1}(z) + be^{-i\phi} \quad (z \in \mathbb{C} - K)$$

is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_{a+b}$.

Also, f satisfies the Laurent series condition of the Flow Mapping Theorem since J_α^{-1} does; see Example 5.1.

(b) A suitable complex potential for the flow past K with circulation $2\pi c$ around K is given by

$$\begin{aligned}
\Omega(z) &= (\Omega_{a+b,c} \circ f)(z) \\
&= \Omega_{a+b,c}(J_\alpha^{-1}(z) + be^{-i\phi}) \\
&= w + be^{-i\phi} + \frac{(a+b)^2}{w+be^{-i\phi}} \\
&\quad - ic\,\mathrm{Log}(w + be^{-i\phi}),
\end{aligned}$$

where $w = J_\alpha^{-1}(z)$, as required.

(c) Next we differentiate Ω to find the velocity function q using the identity

$$(J_\alpha^{-1})'(z) = \left(1 - \frac{\alpha^2}{w^2}\right)^{-1} = \frac{w^2}{w^2-\alpha^2},$$

from Lemma 4.1(c). We obtain

$$\begin{aligned}
q(z) &= \overline{\Omega'(z)} \\
&= \overline{\left(1 - \frac{(a+b)^2}{(w+be^{-i\phi})^2} - \frac{ic}{(w+be^{-i\phi})}\right)} \\
&\quad \times \overline{\left(\frac{w^2}{w^2-\alpha^2}\right)}.
\end{aligned}$$

Remark: A somewhat delicate calculation shows that the limit $\lim\limits_{z\to 2\alpha} q(z)$ of the velocity at the trailing edge of the aerofoil exists if and only if $c = -2(a+b)\sin\phi$, in which case the circulation is negative. For this value of c we have

$$\lim_{z\to 2\alpha} q(z) = \frac{\alpha\cos\phi}{a+b}.$$

Note that for $\phi = 0$ this expression equals $a/(a+b)$, as found in the solution to Exercise 5.1.

Solution to Exercise 5.3

(a) The image under J_a of $\mathbb{C} - K_a$ is $\mathbb{C} - [-2a, 2a]$, so we choose $a = \frac{5}{4}$.

(b) The function $f = J_{5/4}^{-1} \circ J$ is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_{5/4}$, which satisfies the Laurent series condition of the Flow Mapping Theorem, by the hint.

Hence, by the Flow Mapping Theorem, a complex potential function for the flow past this obstacle is

$$\begin{aligned}
\Omega(z) &= (\Omega_{5/4,c} \circ f)(z) \\
&= J(z) - ic\,\mathrm{Log}\big(J_{5/4}^{-1}(J(z))\big),
\end{aligned}$$

by the hint.

Proceeding as in the solution to Exercise 4.5, with $a = 5/4$ instead of 1 and $J(z)$ instead of z, we find that the velocity function is

$$q(z) = \overline{\left(1 - \frac{ic}{J(z)\sqrt{1-4(5/4)^2/(J(z))^2}}\right)J'(z)}$$
$$= \overline{\left(1 - \frac{ic}{J(z)\sqrt{1-(25/4)/(J(z))^2}}\right)J'(z)},$$

as required.

Solution to Exercise 5.4

By the result of Exercise 3.3, the function J_{ia} is a one-to-one conformal mapping from $\mathbb{C} - K_r$ onto $\mathbb{C} - K$, so the function $f = J_{ia}^{-1}$ is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_r$. Also, as in Example 5.1, $f = J_{ia}^{-1}$ satisfies the Laurent series condition of the Flow Mapping Theorem.

Hence, by the Flow Mapping Theorem, a complex potential function for flow past this obstacle is

$$\Omega(z) = (\Omega_{r,c} \circ J_{ia}^{-1})(z)$$
$$= J_{ia}^{-1}(z) + \frac{r^2}{J_{ia}^{-1}(z)} - ic\,\mathrm{Log}(J_{ia}^{-1}(z)),$$

where

$$J_{ia}^{-1}(z) = \tfrac{1}{2}(z + z\sqrt{1+4a^2/z^2}).$$

Then

$$\Omega'(z) = \left(1 - \frac{r^2}{(J_{ia}^{-1}(z))^2} - \frac{ic}{J_{ia}^{-1}(z)}\right)(J_{ia}^{-1})'(z)$$
$$= \left(1 - \frac{r^2}{(J_{ia}^{-1}(z))^2} - \frac{ic}{J_{ia}^{-1}(z)}\right)$$
$$\times \left(\frac{1}{1-(ia)^2/(J_{ia}^{-1}(z))^2}\right)$$
$$= \frac{(J_{ia}^{-1}(z))^2 - r^2 - icJ_{ia}^{-1}(z)}{(J_{ia}^{-1}(z))^2 + a^2}$$
$$= \frac{(J_{ia}^{-1}(z))^2 + a^2 - (r^2+a^2) - icJ_{ia}^{-1}(z)}{(J_{ia}^{-1}(z))^2 + a^2}$$
$$= 1 - \left(\frac{r^2 + a^2 + icJ_{ia}^{-1}(z)}{(J_{ia}^{-1}(z))^2 + a^2}\right),$$

so

$$q(z) = 1 - \overline{\left(\frac{r^2 + a^2 + icJ_{ia}^{-1}(z)}{(J_{ia}^{-1}(z))^2 + a^2}\right)},$$

as required.

Solution to Exercise 5.5

The function J^{-1} maps $\mathbb{C} - K$ onto $\mathbb{C} - D$, and the translation $g(w) = w - \beta$ maps $\mathbb{C} - D$ onto $\mathbb{C} - K_{|1-\beta|}$, where $K_{|1-\beta|}$ is the closed disc with centre 0 and radius $|1-\beta|$. Hence the function $f = g \circ J^{-1}$ is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_{|1-\beta|}$.

Also, f satisfies the Laurent series condition of the Flow Mapping Theorem since J^{-1} does.

Hence, by the Flow Mapping Theorem, a complex potential function for this Obstacle Problem is

$$\Omega(z) = (\Omega_{|1-\beta|,c} \circ f)(z)$$
$$= \Omega_{|1-\beta|,c}(J^{-1}(z) - \beta)$$
$$= w - \beta + \frac{|1-\beta|^2}{w-\beta} - ic\,\mathrm{Log}(w-\beta),$$

where $w = J^{-1}(z)$.

The corresponding velocity function is

$$q(z) = \overline{\left(1 - \frac{|1-\beta|^2}{(w-\beta)^2} - \frac{ic}{w-\beta}\right)\frac{w^2}{w^2-1}},$$

using the identity $(J^{-1})'(z) = 1/(1-1/w^2)$ from Lemma 4.1(c), with $a = 1$.

Unit D2

The Mandelbrot set

Introduction

Many techniques for solving equations involve *iteration*, otherwise known as the method of 'refining guesses'. For example, one way to solve a real polynomial equation $p(x) = 0$ is to use the *Newton–Raphson method*, based on sequences (x_n) defined by the recurrence relation

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}, \quad n = 0, 1, 2, \ldots.$$

The justification of this recurrence relation is shown in Figure 0.1, which indicates that if x_n is 'close' to a zero a of p, then x_{n+1} is (usually) even closer. For example, if $p(x) = x^2 - 2$, then the Newton–Raphson recurrence relation is

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), \quad n = 0, 1, 2, \ldots.$$

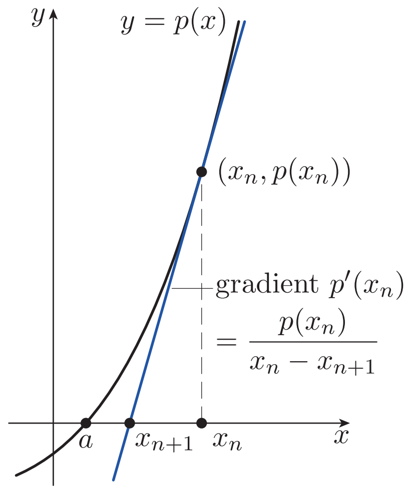


Figure 0.1 The tangent line through the point $(x_n, p(x_n))$ intersects the x-axis at $x = x_{n+1}$

An initial guess of $x_0 = 2$ for a solution of $p(x) = 0$ yields the sequence of refined guesses $2, 1.5, 1.417, \ldots$, which converges rapidly to the known solution $\sqrt{2}$.

In this unit we study the behaviour of *complex* iteration processes of the form

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$

where f is an analytic function. Usually f will be a polynomial function, although we spend a short time in Section 1 discussing the Newton–Raphson method for polynomial functions, which involves the iteration of rational functions. Most of Section 1 is devoted to the basic notions associated with iteration: *nth iterates*, *fixed points* and *conjugate iteration sequences*.

Section 2 takes up the iteration of complex quadratic functions, which is the main topic of the unit. We begin by showing that it is sufficient to consider quadratic functions belonging to the basic family $\{P_c(z) = z^2 + c : c \in \mathbb{C}\}$, because each quadratic function is equivalent, in a certain sense, to one of this form. We then consider, for any given $c \in \mathbb{C}$, the set of points E_c which 'escape to ∞' under iteration of P_c, and establish some basic properties of E_c and its complement K_c. To determine some of the points of K_c, we introduce the idea of a *periodic cycle* of points; we show that certain periodic cycles must lie in the interior of K_c, whereas others lie on the boundary of K_c. This boundary is called the *Julia set* J_c of P_c (see Figure 0.2).

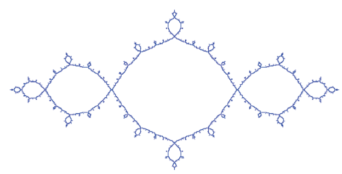


Figure 0.2 A Julia set

In Section 3 we use a technique called *graphical iteration*, which applies only to real functions, to obtain various properties of the set K_c in the case when c is real. For example, we determine the nature of the set $K_c \cap \mathbb{R}$ for all real numbers c.

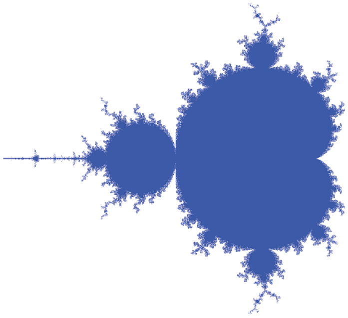


Figure 0.3 Mandelbrot set

In Section 4 we define the *Mandelbrot set*, which is the set M of numbers c such that the set K_c is connected, and we obtain a criterion for c to belong to M. This criterion makes it possible to plot pictures of M (see Figure 0.3) and thus reveal its immensely complicated structure. We investigate this structure by using a result which states that if P_c has a so-called *attracting cycle*, then $c \in M$.

You may be tempted to think that the Mandelbrot set is in some sense an oddity, arising from some special property of quadratic functions. However, in Section 5 we describe briefly some different families of iteration sequences, and we find that the Mandelbrot set also appears in these.

Unit guide

You may find some of the ideas in this unit challenging. However, your labours will be rewarded as you gain some insight into a fascinating subject, which has been and continues to be the object of intense research.

Many of the figures in this unit are shown inside boxes, sometimes labelled with appropriate coordinates, rather than with axes.

Subsections 1.4, 2.4, 4.3 and 4.4, and Section 5, are for reading only – they will not be assessed.

1 Iteration of analytic functions

After working through this section, you should be able to:

- calculate several terms of a given *iteration sequence*
- determine the *nth iterate* of a given analytic function, for small values of n
- determine the *fixed points* of certain analytic functions and find their nature
- calculate *conjugate iteration sequences*
- determine the *Newton–Raphson function* N for a given polynomial function p, and describe the iterative behaviour of N when p is a quadratic function.

1.1 Defining an iteration sequence

Any sequence (z_n) defined by a recurrence relation of the form

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$

where f is a function, is called an **iteration sequence** with **initial term** z_0. Note that some texts call z_0 the *seed* and (z_n) the *orbit* of z_0, and refer to an iteration sequence as a *dynamical system*.

For example, if $f(z) = 2z$, then the recurrence relation is

$$z_{n+1} = 2z_n, \quad n = 0, 1, 2, \ldots.$$

If $z_0 = i$, then the first few terms of the corresponding iteration sequence are

$$z_0 = i, \quad z_1 = 2i, \quad z_2 = 4i, \quad z_3 = 8i, \quad \ldots .$$

In this case the sequence (z_n) tends to infinity, but if we had chosen $z_0 = 0$, then the corresponding iteration sequence would be constant:

$$z_0 = 0, \quad z_1 = 0, \quad z_2 = 0, \quad z_3 = 0, \quad \ldots .$$

Thus the behaviour of a given iteration sequence depends not only on the function f, but also on the choice of initial term z_0. We often represent such iteration sequences, as in Figure 1.1, by plotting the points $z_0, z_1, \ldots$, and indicating how these points are related by the function f.

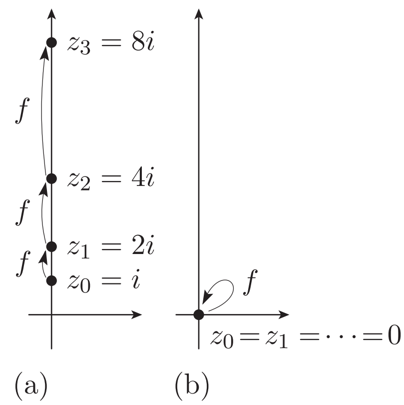


Figure 1.1 The sequence $z_{n+1} = 2z_n$, $n = 0, 1, \ldots$, with: (a) $z_0 = i$, (b) $z_0 = 0$

Exercise 1.1

Calculate and plot the terms z_0, z_1, z_2, z_3 for each of the following iteration sequences, and write down the corresponding functions f.

(a) $z_{n+1} = z_n^2, \quad z_0 = i$ (b) $z_{n+1} = \frac{1}{2}z_n + 1, \quad z_0 = 0$

(c) $z_{n+1} = z_n^2 - 1, \quad z_0 = 0$ (d) $z_{n+1} = z_n^2 + i, \quad z_0 = 0$

Consider the sequence (z_n) defined by

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots .$$

Then

$$z_1 = f(z_0), \quad z_2 = f(z_1) = f(f(z_0)), \quad z_3 = f(z_2) = f(f(f(z_0))),$$

and, in general,

$$z_n = f(f(\cdots(f(z_0))\cdots)), \quad \text{for } n = 1, 2, \ldots, \tag{1.1}$$

where the function f is applied n times. We introduce a notation for such repeated compositions.

Definition

Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a function. The ***n*th iterate** of f is the function f^n obtained by applying f exactly n times:

$$f^n = f \circ f \circ \cdots \circ f, \quad \text{where } n = 1, 2, \ldots .$$

Also, f^0 denotes the identity function $f^0(z) = z$.

For example,

$$f^1(z) = f(z) \quad \text{and} \quad f^2(z) = f(f(z)).$$

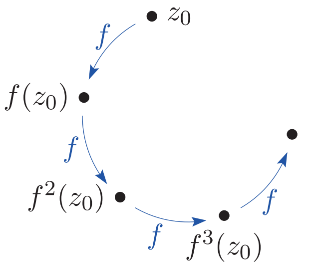


Figure 1.2 Images of z_0 under iterates of f

Remarks

1. Note that we write $f^2(z)$ to mean $f(f(z))$, which is different from $(f(z))^2$. However, when using trigonometric functions we continue to write $\sin^2 z$ for $(\sin z)^2$, for example, since this is standard practice in mathematics.
2. Equation (1.1) can be written in the convenient form
$$z_n = f^n(z_0), \quad n = 1, 2, \ldots;$$
see Figure 1.2.
3. Note that if $m, n \in \mathbb{N} \cup \{0\}$, then
$$f^m(f^n(z)) = f^{m+n}(z) = f^n(f^m(z)),$$
since composition of functions is associative.

Example 1.1

Determine the rules for the functions f^2 and f^3 when $f(z) = z^2 - 1$.

Solution

We have
$$\begin{aligned} f^2(z) &= f(f(z)) \\ &= f(z^2 - 1) \\ &= (z^2 - 1)^2 - 1 \\ &= z^4 - 2z^2, \end{aligned}$$
so
$$\begin{aligned} f^3(z) &= f(f^2(z)) \\ &= (z^4 - 2z^2)^2 - 1 \\ &= z^8 - 4z^6 + 4z^4 - 1. \end{aligned}$$
Alternatively,
$$\begin{aligned} f^3(z) &= f^2(f(z)) \\ &= (z^2 - 1)^4 - 2(z^2 - 1)^2 \\ &= (z^8 - 4z^6 + 6z^4 - 4z^2 + 1) - 2(z^4 - 2z^2 + 1) \\ &= z^8 - 4z^6 + 4z^4 - 1. \end{aligned}$$

Exercise 1.2

Determine the rules for the functions f^2 and f^3 when $f(z) = \frac{1}{2}z + 1$.

The solution to Example 1.1 suggests that for some functions f it may be difficult to find a general formula for the nth iterate f^n. However, there are a few simple cases for which formulas can be found.

Example 1.2

Find a formula for the nth iterate f^n of each of the following functions.

(a) $f(z) = az$, where $a \in \mathbb{C}$ (b) $f(z) = z^2$

Solution

(a) We have

$$\begin{aligned} f^1(z) &= f(z) = az, \\ f^2(z) &= f(f(z)) = f(az) = a(az) = a^2z, \\ f^3(z) &= f(f^2(z)) = f(a^2z) = a(a^2z) = a^3z, \end{aligned}$$

and, in general, by the Principle of Mathematical Induction,

$$f^n(z) = a^n z, \quad \text{for } n = 1, 2, \ldots.$$

(b) We have

$$\begin{aligned} f^1(z) &= f(z) = z^2, \\ f^2(z) &= f(f(z)) = f(z^2) = (z^2)^2 = z^4, \\ f^3(z) &= f(f^2(z)) = f(z^4) = (z^4)^2 = z^8, \end{aligned}$$

and, in general,

$$f^n(z) = z^{2^n}, \quad \text{for } n = 1, 2, \ldots.$$

The following exercise gives some other functions for which we can find explicit formulas for the iterates.

Exercise 1.3

Find a formula for the nth iterate f^n of each of the following functions.

(a) $f(z) = z + b$, where $b \in \mathbb{C}$ (b) $f(z) = z^3$

The formulas obtained in Example 1.2 and Exercise 1.3 can be used to determine the behaviour of the corresponding iteration sequences, as we now illustrate.

Example 1.3

(a) Prove that if $f(z) = az$, where $a \in \mathbb{C}$, $|a| < 1$ and $z_0 \in \mathbb{C}$, then

$$z_n = f^n(z_0) \to 0 \text{ as } n \to \infty.$$

(b) Prove that if $f(z) = z^2$ and $|z_0| < 1$, then

$$z_n = f^n(z_0) \to 0 \text{ as } n \to \infty.$$

(c) Prove that if $f(z) = z + b$, where $b \in \mathbb{C}$, $b \neq 0$ and $z_0 \in \mathbb{C}$, then

$$z_n = f^n(z_0) \to \infty \text{ as } n \to \infty.$$

Solution

(a) By Example 1.2(a),

$$z_n = f^n(z_0) = a^n z_0, \quad \text{for } n = 1, 2, \ldots.$$

Since $|a| < 1$, we know that (a^n) is a basic null sequence, so (z_n) is a null sequence (see Theorems 1.2 and 1.3 of Unit A3).

(b) By Example 1.2(b),

$$z_n = f^n(z_0) = z_0^{2^n}, \quad \text{for } n = 1, 2, \ldots.$$

Next we use the Binomial Theorem to show that

$$2^n = (1+1)^n = 1 + n + \cdots \geq n, \quad \text{for } n = 1, 2, \ldots.$$

Hence

$$|z_n| = |z_0|^{2^n} \leq |z_0|^n, \quad \text{for } n = 1, 2, \ldots.$$

Since $|z_0| < 1$, $(|z_0|^n)$ is a null sequence and therefore so is (z_n), by the Squeeze Rule (Theorem 1.1 of Unit A3).

(c) By Exercise 1.3(a),

$$z_n = f^n(z_0) = z_0 + nb, \quad \text{for } n = 1, 2, \ldots.$$

Thus

$$\begin{aligned}\frac{1}{z_n} &= \frac{1}{z_0 + nb} \\ &= \frac{1/n}{z_0/n + b} \\ &\to \frac{0}{b} = 0 \text{ as } n \to \infty \text{ (since } b \neq 0).\end{aligned}$$

Hence, by the Reciprocal Rule (Theorem 1.5 of Unit A3),

$$z_n \to \infty \text{ as } n \to \infty.$$

In Example 1.3(a) we saw that $f^n(z_0) \to 0$ as $n \to \infty$ for all choices of initial term z_0, whereas in Example 1.3(b) we saw that $f^n(z_0) \to 0$ as $n \to \infty$ whenever $|z_0| < 1$. In the next exercise we ask you to investigate what happens for this latter function f with other initial values.

Exercise 1.4

Let $f(z) = z^2$. Determine the behaviour of the iteration sequence

$$z_n = f^n(z_0), \quad n = 1, 2, \ldots,$$

when

(a) $z_0 = 1$, (b) $z_0 = -i$, (c) $z_0 = e^{2\pi i/3}$, (d) $|z_0| > 1$.

1.2 Fixed points

Whenever an iteration sequence defined by a *continuous* function f converges to a limit α, say, then the point α has the property that $f(\alpha) = \alpha$, as we now show.

Suppose that the iteration sequence $z_n = f^n(z_0)$ satisfies

$$z_n \to \alpha \text{ as } n \to \infty.$$

Then the sequence (z_{n+1}), which is just the sequence (z_n) with its first term removed, also has the property that $z_{n+1} \to \alpha$ as $n \to \infty$. Therefore

$$\alpha = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} f(z_n) = f(\alpha),$$

the last equality holding because the function f is continuous at α. As the limit α satisfies $f(\alpha) = \alpha$, it is called a *fixed point* of the function f; see Figure 1.3.

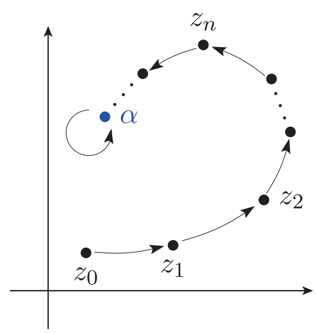


Figure 1.3 A fixed point α

Definition

A **fixed point** of a function f is a point α for which $f(\alpha) = \alpha$.

For example, the function $f(z) = 2z$ has 0 as a fixed point, and the function $f(z) = z^2$ has 0 and 1 as fixed points. In general, we find the fixed points (if any) of a given function f by solving the **fixed point equation** $f(z) = z$.

Exercise 1.5

Determine the fixed points of each of the following functions.

(a) $f(z) = \frac{1}{2}z + 1$ (b) $f(z) = z^2 - 2$ (c) $f(z) = z^3$

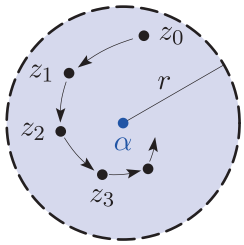


Figure 1.4 An iteration sequence attracted to a fixed point

The behaviour of an iteration sequence $z_n = f^n(z_0)$, $n = 1, 2, \ldots$, near a fixed point α of an *analytic* function f depends to a great extent on the derivative of f at α. Using the idea of a complex scale factor from Subsection 1.5 of Unit A4, we see that if $|f'(\alpha)| < 1$, then to a good approximation f maps any small disc with centre α to an even smaller disc with centre α. Thus an initial term z_0 near α gives rise to an iteration sequence which is attracted to α (see Figure 1.4).

Theorem 1.1

Let α be a fixed point of an analytic function f, and suppose that $|f'(\alpha)| < 1$. Then there exists $r > 0$ such that

$$\lim_{n\to\infty} f^n(z_0) = \alpha, \quad \text{for } |z_0 - \alpha| < r.$$

Proof We first choose a real number c such that $|f'(\alpha)| < c < 1$. Since

$$f'(\alpha) = \lim_{z\to\alpha} \frac{f(z) - f(\alpha)}{z - \alpha} \quad \text{and} \quad c - |f'(\alpha)| > 0,$$

there is a positive number r such that

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} - f'(\alpha) \right| < c - |f'(\alpha)|, \quad \text{for } 0 < |z - \alpha| < r.$$

This follows from the ε-δ definition of limit (Subsection 3.2 of Unit A3) by putting $\varepsilon = c - |f'(\alpha)|$ and $\delta = r$.

Hence, by the Triangle Inequality,

$$\begin{aligned}\left| \frac{f(z) - f(\alpha)}{z - \alpha} \right| &\le \left| \frac{f(z) - f(\alpha)}{z - \alpha} - f'(\alpha) \right| + |f'(\alpha)| \\ &\le c - |f'(\alpha)| + |f'(\alpha)| = c,\end{aligned}$$

for $0 < |z - \alpha| < r$. Thus, since $f(\alpha) = \alpha$, we have

$$|f(z) - \alpha| \le c|z - \alpha|, \quad \text{for } |z - \alpha| < r.$$

In particular, if $|z_0 - \alpha| < r$, then

$$|f(z_0) - \alpha| \le c|z_0 - \alpha| < cr < r,$$

so $|f(z_0) - \alpha| < r$, also.

Hence, by repeating this process, we obtain

$$|f^2(z_0) - \alpha| \le c|f(z_0) - \alpha| \le c^2|z_0 - \alpha| < c^2 r < r,$$

and, in general,

$$|f^n(z_0) - \alpha| \le c^n|z_0 - \alpha|, \quad \text{for } n = 1, 2, \ldots. \tag{1.2}$$

Since $0 < c < 1$, the sequence (c^n) is a null sequence, so

$$\lim_{n\to\infty} f^n(z_0) = \alpha, \quad \text{for } |z_0 - \alpha| < r. \qquad \blacksquare$$

Notice in the proof of Theorem 1.1 that the smaller $|f'(\alpha)|$ is, the smaller we can choose the number c to be (such that $|f'(\alpha)| < c < 1$) and hence the faster the sequence $(f^n(z_0))$ converges to α, by inequality (1.2). This convergence is especially fast if $f'(\alpha) = 0$.

If α is a fixed point of f for which $|f'(\alpha)| > 1$, then we should expect an initial term z_0 near α (but not α itself) to be pushed away from α by f. If $|f'(\alpha)| = 1$, then the behaviour depends in a more subtle way on the precise value of $f'(\alpha)$. These observations suggest the following classification of fixed points.

Definitions

A fixed point α of an analytic function f is

- **attracting** if $|f'(\alpha)| < 1$
- **repelling** if $|f'(\alpha)| > 1$
- **indifferent** if $|f'(\alpha)| = 1$
- **super-attracting** if $f'(\alpha) = 0$.

Remarks

1. Note that a super-attracting fixed point is a special type of attracting fixed point.
2. Some texts use *attractive* or *stable* instead of attracting; *repulsive* or *unstable* instead of repelling; and *neutral* instead of indifferent.

As an example, the function $f(z) = az$, where $a \in \mathbb{C}$, has 0 as a fixed point, and since $f'(z) = a$, this fixed point is attracting if $|a| < 1$, repelling if $|a| > 1$ and indifferent if $|a| = 1$.

Exercise 1.6

For each of the following functions f, classify the given fixed points α as attracting, repelling or indifferent, and identify any attracting fixed points that are super-attracting.

(a) $f(z) = z^2, \quad \alpha = 0, 1$

(b) $f(z) = \frac{1}{2}z + 1, \quad \alpha = 2$

(c) $f(z) = z^2 - 2, \quad \alpha = 2$

For any given analytic function f with an attracting fixed point α, it is natural to ask exactly which points z are attracted to α under iteration of f (that is, $f^n(z) \to \alpha$ as $n \to \infty$), so we make the following definition. Here, and subsequently, we may use z, rather than z_0, as an initial term when we do not need to label the sequence (z_n).

Definition

Let α be an attracting fixed point of an analytic function f. Then the **basin of attraction** of α under f is the set

$$\{z : f^n(z) \to \alpha \text{ as } n \to \infty\}.$$

A simple example is the basin of attraction of $\alpha = 0$ under the function $f(z) = z^2$. This is the open unit disc because

$$f^n(z_0) \to 0 \text{ as } n \to \infty, \quad \text{for } |z_0| < 1,$$

by Example 1.3(b), but

$$f^n(z_0) \not\to 0 \text{ as } n \to \infty, \quad \text{for } |z_0| \geq 1,$$

since $|f^n(z_0)| = |z_0|^{2^n} \geq 1$, for $n = 1, 2, \ldots$. Later in the unit we will see much more complicated examples of basins of attraction.

Exercise 1.7

For each of the following functions f, determine the basin of attraction of the given fixed point α.

(a) $f(z) = \frac{1}{2}z, \quad \alpha = 0$ (b) $f(z) = z^3, \quad \alpha = 0$

1.3 Conjugate iteration sequences

Consider the iteration sequence

$$z_{n+1} = z_n^2 + 2z_n, \quad n = 0, 1, 2, \ldots, \text{ with } z_0 = -\tfrac{1}{2}, \tag{1.3}$$

and suppose that we wish to find a formula for z_n in terms of n. This iteration sequence is quite complicated, so it is sensible to look first at a few terms of (z_n):

$$z_0 = -\tfrac{1}{2}, \quad z_1 = -\tfrac{3}{4}, \quad z_2 = -\tfrac{15}{16}, \quad z_3 = -\tfrac{255}{256}, \quad \ldots.$$

It appears that $z_n \to -1$ as $n \to \infty$, which suggests that we should make the substitution

$$z_n = w_n - 1,$$

and try to find a formula for w_n. Substituting $z_n = w_n - 1$ and $z_{n+1} = w_{n+1} - 1$ in equation (1.3), we obtain

$$\begin{aligned} w_{n+1} - 1 &= (w_n - 1)^2 + 2(w_n - 1) \\ &= w_n^2 - 2w_n + 1 + 2w_n - 2 \\ &= w_n^2 - 1; \end{aligned}$$

hence

$$w_{n+1} = w_n^2.$$

The iteration sequence

$$w_{n+1} = w_n^2, \quad n = 0, 1, 2, \ldots, \text{ with } w_0 = z_0 + 1 = \tfrac{1}{2}, \tag{1.4}$$

is simpler than the one given in equation (1.3) and, moreover, we know from Example 1.2(b) that

$$w_n = \left(\tfrac{1}{2}\right)^{2^n}, \quad \text{for } n = 0, 1, 2, \ldots.$$

We deduce that the formula for z_n in terms of n is

$$z_n = \left(\tfrac{1}{2}\right)^{2^n} - 1, \quad \text{for } n = 0, 1, 2, \ldots.$$

More generally, suppose that we wish to investigate the iteration sequence

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$

and h is a one-to-one function. Then we claim that

$$w_n = h(z_n), \quad n = 0, 1, 2, \ldots,$$

is also an iteration sequence. Indeed, the inverse function h^{-1} exists because h is one-to-one, so $z_n = h^{-1}(w_n)$ and

$$\begin{aligned} w_{n+1} &= h(z_{n+1}) \\ &= h(f(z_n)) \\ &= h\big(f(h^{-1}(w_n))\big), \quad \text{for } n = 0, 1, 2, \ldots. \end{aligned}$$

Thus

$$w_{n+1} = g(w_n), \quad \text{for } n = 0, 1, 2, \ldots,$$

where the function g is given by $g = h \circ f \circ h^{-1}$ (see Figure 1.5). With a suitable choice of h, the function g may be simpler than f (as in the example above, where $f(z) = z^2 + 2z$, $h(z) = z + 1$ and $g(w) = w^2$).

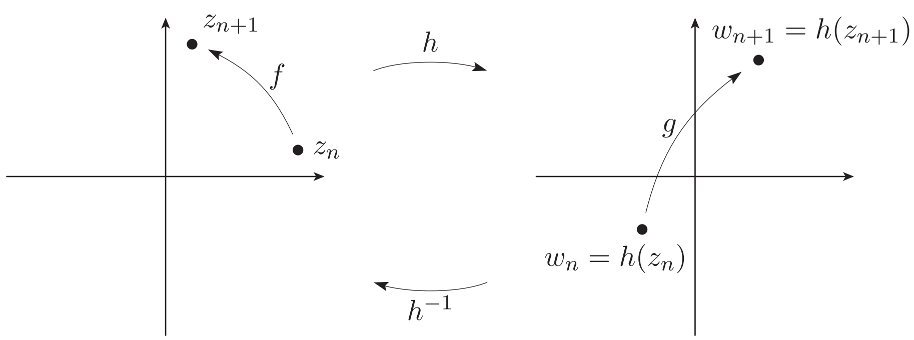


Figure 1.5 Conjugate functions f and $g = h \circ f \circ h^{-1}$

We make the following definitions.

Definitions

The functions f and g are **conjugate** to each other if

$$g = h \circ f \circ h^{-1},$$

for some one-to-one function h called the **conjugating function**.

Let (z_n) be the sequence defined by

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$

for some initial term z_0, and let $w_n = h(z_n)$, for $n = 0, 1, 2, \ldots$. Then the sequence (w_n) satisfies

$$w_{n+1} = g(w_n), \quad \text{for } n = 0, 1, 2, \ldots,$$

and (z_n) and (w_n) are called **conjugate iteration sequences**.

Remarks

1. Do not confuse this use of the word 'conjugate', which is borrowed from group theory, with 'complex conjugate'.
2. In practice the function g is usually found by substituting $z_n = h^{-1}(w_n)$ and $z_{n+1} = h^{-1}(w_{n+1})$ into $z_{n+1} = f(z_n)$ and then rearranging, as we did with equation (1.3).

Since the sequence (w_n) is the image of the sequence (z_n) under the function h, it follows that knowing the behaviour of one sequence may help us to understand the behaviour of the other. For example, if both h and h^{-1} are continuous functions, then (z_n) is convergent if and only if (w_n) is convergent. Also, α is a fixed point of f if and only if $h(\alpha)$ is a fixed point of g.

If the conjugating function h is required to be one-to-one *and* entire, then it must in fact be of the form $h(z) = az + b$, where $a \neq 0$. This is because any polynomial of degree at least 2 is not one-to-one, and if h is entire but not a polynomial, then $h(1/z)$ has an essential singularity at 0, in which case the Casorati–Weierstrass Theorem (Theorem 3.3 of Unit B4) implies that $h(1/z)$ is not one-to-one on $\mathbb{C} - \{0\}$, so h is not one-to-one on $\mathbb{C}$.

Exercise 1.8

Prove that the iteration sequence

$$z_{n+1} = z_n - z_n^2, \quad n = 0, 1, 2, \ldots,$$

is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + \tfrac{1}{4}, \quad n = 0, 1, 2, \ldots,$$

with conjugating function $h(z) = -z + \frac{1}{2}$. Determine w_0 when $z_0 = \frac{1}{2}$.

Exercise 1.9

(a) Prove that the iteration sequence

$$z_{n+1} = az_n + b, \quad n = 0, 1, 2, \ldots,$$

where $a \neq 1$, is conjugate to the iteration sequence

$$w_{n+1} = aw_n, \quad n = 0, 1, 2, \ldots,$$

with conjugating function $h(z) = z + b/(a-1)$.

(Note that you considered the case $a = 1$ in Exercise 1.3(a).)

(b) Hence obtain a formula for z_n in terms of z_0, and describe the behaviour of the sequence (z_n) when

(i) $|a| < 1$, (ii) $|a| = 1$, $a \neq 1$, (iii) $|a| > 1$.

1.4 The Newton–Raphson method

This subsection is intended for reading only (it will not be assessed), and Exercises 1.10 and 1.11 are optional.

We will now see how the ideas introduced so far help with analysing the Newton–Raphson method, described briefly in the Introduction. Let p be a complex polynomial function. Then the corresponding Newton–Raphson iteration sequence is

$$z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)}, \quad n = 0, 1, 2, \ldots.$$

This recurrence relation shows that (z_n) is an iteration sequence obtained by iterating the function

$$N(z) = z - \frac{p(z)}{p'(z)},$$

which is known as the **Newton–Raphson function** of p. In general, N is a rational function (unless p is of degree 1). If $p'(\alpha) = 0$, then we define $N(\alpha) = \infty$. Also, we can extend N to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in the manner described in Subsection 1.3 of Unit C3. Thus a Newton–Raphson iteration sequence may include ∞ among its terms.

A key observation is that if α is a *simple* zero of p, then $p'(\alpha) \neq 0$ (see Subsection 5.1 of Unit B3), so $N(\alpha) = \alpha$. Thus α is a fixed point of N. To classify it, we evaluate $N'(\alpha)$:

$$\begin{aligned} N'(\alpha) &= 1 - \frac{(p'(\alpha))^2 - p(\alpha)p''(\alpha)}{(p'(\alpha))^2} \\ &= 1 - \frac{(p'(\alpha))^2}{(p'(\alpha))^2} \quad (\text{since } p(\alpha) = 0) \\ &= 0. \end{aligned}$$

Thus a simple zero α of p is a super-attracting fixed point for the Newton–Raphson function N. This is good news because it implies that the Newton–Raphson method always converges rapidly to a simple zero α of p provided that our initial guess is close enough to α.

If α is a zero of p of order k, where $k > 1$, then it can be shown that α is a fixed point of the function N such that

$$N'(\alpha) = 1 - 1/k < 1,$$

so the Newton–Raphson method still works, but not so well; you can investigate this if you wish in Exercise 1.11.

In 1879 the English mathematician Arthur Cayley (1821–1895), who is well known for his pioneering work in group theory and linear algebra, decided to analyse the Newton–Raphson method when p is the quadratic function

$$p(z) = z^2 + az + b, \quad \text{where } a, b \in \mathbb{C}.$$

Suppose that the function p has distinct zeros at α and β, say. Then these zeros must be simple, so α and β are super-attracting fixed points of the Newton–Raphson function N. By Theorem 1.1, there are open discs centred at α and β whose points are attracted to α and β, respectively, under iteration of N.

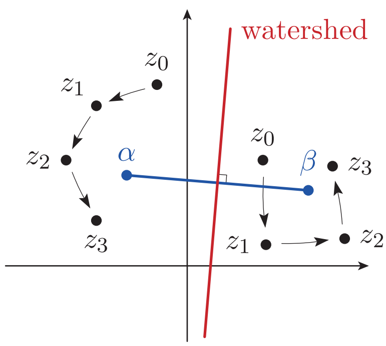


Figure 1.6 Basins of attraction of α and β

Cayley wanted to know which points in $\mathbb{C}$ are attracted to α under iteration of N, and which are attracted to β. In other words, what are the basins of attraction of α and β under N? Cayley found that the answer is remarkably simple: the perpendicular bisector of the line segment joining α and β forms a 'watershed' for this iteration process (see Figure 1.6). If z_0 falls on the same side of the watershed as α, then $N^n(z_0) \to \alpha$ as $n \to \infty$, but if z_0 falls on the other side, then $N^n(z_0) \to \beta$ as $n \to \infty$. If z_0 falls exactly on the watershed line, then the sequence $(N^n(z_0))$ remains on the line!

The Newton–Raphson function for $p(z) = z^2 + az + b$ is

$$\begin{aligned} N(z) &= z - \frac{p(z)}{p'(z)} \\ &= z - \frac{z^2 + az + b}{2z + a} \\ &= \frac{z^2 - b}{2z + a}, \end{aligned}$$

but from this it is not at all evident why Cayley's result should be true. However, we obtain a much simpler iteration sequence by using the conjugating function

$$h(z) = \frac{z - \alpha}{z - \beta}.$$

This function h is a Möbius transformation, which, when using the conventions of Unit C3 (see Subsection 2.1 of Unit C3), is a one-to-one function from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$, and maps α to 0 and β to ∞. Also, h has the remarkable property that

$$h(N(z)) = (h(z))^2 \tag{1.5}$$

(see Exercise 1.10).

Now define $g(w) = w^2$. Then equation (1.5) is equivalent to the equation $g(z) = h\big(N\big(h^{-1}(z)\big)\big)$, that is, $g = h \circ N \circ h^{-1}$, which implies that

$$z_{n+1} = N(z_n), \quad n = 0, 1, 2, \ldots,$$

and

$$w_{n+1} = g(w_n) = w_n^2, \quad n = 0, 1, 2, \ldots,$$

are conjugate iteration sequences with conjugating function h.

Now, we know from Example 1.3(b) that

$$w_n \to 0 \text{ as } n \to \infty, \quad \text{if } |w_0| < 1, \tag{1.6}$$

and from Exercise 1.4(d) that

$$w_n \to \infty \text{ as } n \to \infty, \quad \text{if } |w_0| > 1, \tag{1.7}$$

and it is clear that the sequence (w_n) remains on the circle $\{w : |w| = 1\}$ if $|w_0| = 1$.

To find out what happens to the original sequence (z_n), we note that

$$h^{-1}(w_n) = z_n, \quad h^{-1}(0) = \alpha, \quad h^{-1}(\infty) = \beta,$$

and

$$|w_0| = |h(z_0)| = \left|\frac{z_0 - \alpha}{z_0 - \beta}\right|.$$

Therefore we deduce from properties (1.6) and (1.7) that

$$z_n \to \alpha \text{ as } n \to \infty, \quad \text{if } |z_0 - \alpha| < |z_0 - \beta|,$$

that

$$z_n \to \beta \text{ as } n \to \infty, \quad \text{if } |z_0 - \alpha| > |z_0 - \beta|,$$

and also that (z_n) remains on the extended line

$$\{z : |z - \alpha| = |z - \beta|\} \cup \{\infty\}$$

if z_0 is on this line.

This is Cayley's remarkable solution.

Exercise 1.10

Prove that $h(N(z)) = (h(z))^2$, where

$$h(z) = \frac{z - \alpha}{z - \beta} \quad \text{and} \quad N(z) = \frac{z^2 - b}{2z + a}.$$

(*Hint*: Note that $z = \alpha$ and $z = \beta$ satisfy the equation $z^2 = -(az + b)$.)

Exercise 1.11

Describe what happens under iteration of the function N if $\alpha = \beta$.

(*Hint*: Use the result of Exercise 1.9(b)(i).)

Finally, we look briefly at the Newton–Raphson method for cubic polynomial functions. Here you might guess that the complex plane divides itself into three simple regions, each surrounding one zero of p and consisting of those points that are attracted to that zero under iteration of the Newton–Raphson function N. This seems to have been Cayley's hunch in 1879, although he was unable to prove such a result. With the help of computer-generated pictures we can now see why Cayley had no chance of finding a simple solution to this problem!

Consider $p(z) = z^3 - 1$, whose zeros are

$$\alpha_1 = 1, \quad \alpha_2 = e^{2\pi i/3} = \tfrac{1}{2}(-1 + i\sqrt{3}), \quad \alpha_3 = e^{4\pi i/3} = \tfrac{1}{2}(-1 - i\sqrt{3}).$$

In this case,

$$\begin{aligned} N(z) &= z - \frac{z^3 - 1}{3z^2} \\ &= \frac{2z^3 + 1}{3z^2}, \end{aligned}$$

and $\alpha_1, \alpha_2, \alpha_3$ are super-attracting fixed points of N, since they are simple zeros of p. Figure 1.7 shows in white the basin of attraction of the fixed point $\alpha_1 = 1$.

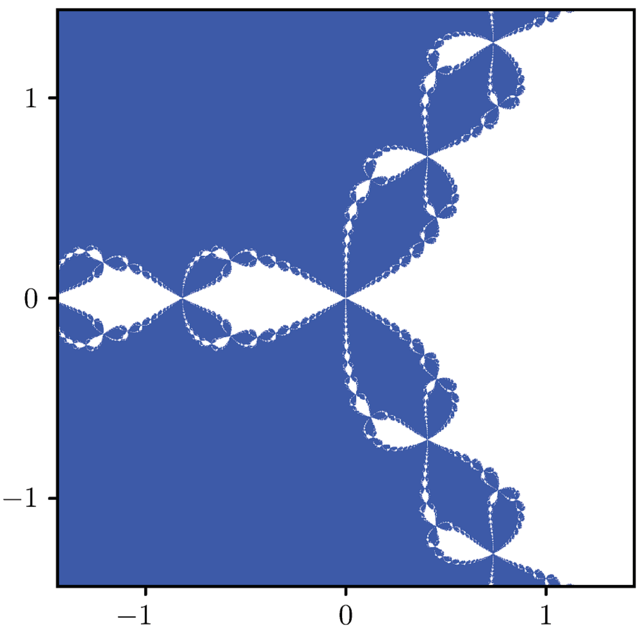


Figure 1.7 The basin of attraction of 1 for N, in white

The basins of attraction of $\alpha_1, \alpha_2, \alpha_3$ must be congruent to each other under rotation about 0 through $2\pi/3$ because of the symmetry of $\alpha_1, \alpha_2, \alpha_3$. But these basins are not at all simple, and they are not even regions (because they are not connected). The union of these three strange basins is almost the whole of $\mathbb{C}$. In addition, there is a complicated 'watershed' which separates the basins of attraction (see Figure 1.8) and which manages, somehow, to be the boundary of all three sets simultaneously!

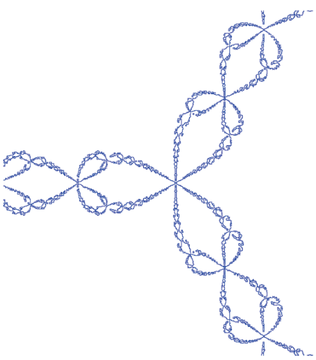


Figure 1.8 The watershed for Newton–Raphson iteration

Thus the iteration of even fairly simple rational functions can lead to very complicated behaviour. In the next three sections we find that such behaviour can occur even for the iteration of simple polynomial functions, and we return to the Newton–Raphson method in Section 5.

Further exercises

Exercise 1.12

Calculate and plot the terms z_0, z_1, z_2, z_3 for each of the following iteration sequences, and write down the corresponding function f.

(a) $z_{n+1} = iz_n, \quad z_0 = 2i$ (b) $z_{n+1} = 2z_n(1 - z_n), \quad z_0 = \frac{1}{4}$

(c) $z_{n+1} = \dfrac{z_n}{z_n + 1}, \quad z_0 = \frac{1}{2}(-1 + i)$

Exercise 1.13

Find all the fixed points of each of the following functions, and classify them as attracting, repelling or indifferent, identifying any attracting fixed points that are super-attracting.

(a) $f(z) = z - z^2$ (b) $f(z) = 2z(1 - z)$ (c) $f(z) = z^2 - \frac{1}{2}$

(d) $f(z) = \dfrac{z}{z + 1}$

Exercise 1.14

Prove that any iteration sequence of the form

$$z_{n+1} = 2z_n(1 - z_n), \quad n = 0, 1, 2, \ldots,$$

is conjugate, using the conjugating function $h(z) = 1 - 2z$, to one of the form

$$w_{n+1} = w_n^2, \quad n = 0, 1, 2, \ldots.$$

Hence obtain a formula for z_n in terms of z_0.

2 Iterating complex quadratics

After working through this section, you should be able to:

- conjugate a given quadratic iteration sequence to one determined by a member of the family of functions $\{P_c : c \in \mathbb{C}\}$, where $P_c(z) = z^2 + c$
- explain why $P_c^n(z_0) \to \infty$ as $n \to \infty$ if $|z_0|$ is large enough
- describe some of the properties of the *escape set* E_c of P_c, and of its complement, the *keep set* K_c
- find the *periodic points* of P_c, for certain values of c, and classify their type
- understand the definition of the *Julia set* J_c.

2.1 The basic quadratic family

In Section 1 we saw that given any iteration sequence of the form

$$z_{n+1} = az_n + b, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

where $a, b \in \mathbb{C}$, we can describe the behaviour of (z_n) for any initial term z_0. In this section we begin to study iteration sequences of the form

$$z_{n+1} = az_n^2 + bz_n + c, \quad n = 0, 1, 2, \ldots, \tag{2.2}$$

where $a, b, c \in \mathbb{C}$ and $a \neq 0$. We will find that the family of such sequences is much more diverse than that given by (2.1). To begin with we note that every iteration sequence of the form (2.2) is in fact conjugate to one of a simpler type.

Theorem 2.1

The iteration sequence

$$z_{n+1} = az_n^2 + bz_n + c, \quad n = 0, 1, 2, \ldots,$$

where $a \neq 0$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2$. The conjugating function is

$$h(z) = az + \tfrac{1}{2}b.$$

Proof We multiply the recurrence relation (2.2) by a and complete the square to obtain

$$az_{n+1} = \left(az_n + \tfrac{1}{2}b\right)^2 + ac - \tfrac{1}{4}b^2, \quad n = 0, 1, 2, \ldots.$$

Thus, putting $w_n = h(z_n)$, for $n = 0, 1, 2, \ldots$, where $h(z) = az + \frac{1}{2}b$, we obtain

$$w_{n+1} - \tfrac{1}{2}b = w_n^2 + ac - \tfrac{1}{4}b^2, \quad n = 0, 1, 2, \ldots;$$

that is,

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2$, as required. ■

There are many different iteration sequences of the form (2.2) that are conjugate to any one iteration sequence of the form $w_{n+1} = w_n^2 + d$. This is illustrated in the following exercise.

Exercise 2.1

Use Theorem 2.1 to show that each of the following iteration sequences is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 - 2, \quad n = 0, 1, 2, \ldots, \text{ with } w_0 = 0.$$

(a) $z_{n+1} = 4z_n(1 - z_n), \quad n = 0, 1, 2, \ldots,$ with $z_0 = \frac{1}{2}$

(b) $z_{n+1} = 1 - 2z_n^2, \quad n = 0, 1, 2, \ldots,$ with $z_0 = 0$

Theorem 2.1 tells us that if we wish to understand the possible behaviour of quadratic functions under iteration, then it is sufficient to consider only those of the form $w_{n+1} = w_n^2 + d$, and it is convenient to relabel these as

$$z_{n+1} = z_n^2 + c, \quad n = 0, 1, 2, \ldots,$$

where c is a complex parameter. We will devote most of the rest of this unit to such iteration sequences, and so we introduce a name for the corresponding quadratic functions.

Definition

Functions of the form

$$P_c(z) = z^2 + c, \quad \text{where } c \in \mathbb{C},$$

are called **basic quadratic functions**.

For example, $P_0(z) = z^2$, $P_1(z) = z^2 + 1$ and $P_i(z) = z^2 + i$ are all basic quadratic functions. In the following exercises we ask you to establish some elementary properties of the basic quadratic functions.

Exercise 2.2

(a) Determine the rules for the functions P_c^2 and P_c^3. (Recall that P_c^n denotes the nth iterate of P_c.)

(b) Write down a formula for $P_c^{n+1}(z)$ in terms of $P_c^n(z)$, and hence prove that P_c^n is an even polynomial function of degree 2^n.

Exercise 2.3

(a) Show that the fixed points of P_c are $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, and prove that if $c \neq \frac{1}{4}$, then at least one of these fixed points is repelling.

(*Hint*: Recall that $\sqrt{z}$ denotes the principal square root of z.)

(b) Determine how many fixed points P_c has when $c = \frac{1}{4}$, and classify their type.

2.2 The escape set and the keep set

In Example 1.2(b) we found the iterates of the function $P_0(z) = z^2$. We saw that the iteration sequence

$$z_n = P_0^n(z_0), \quad n = 0, 1, 2, \ldots,$$

can be determined explicitly as

$$z_n = z_0^{2^n}, \quad \text{for } n = 0, 1, 2, \ldots.$$

Thus

$$\left.\begin{array}{ll} z_n \to 0 \text{ as } n \to \infty, & \text{if } |z_0| < 1, \\ z_n \to \infty \text{ as } n \to \infty, & \text{if } |z_0| > 1, \\ |z_n| = 1, \text{ for } n = 1, 2, \ldots, & \text{if } |z_0| = 1. \end{array}\right\} \tag{2.3}$$

It is natural to ask whether similar behaviour occurs for other values of c. We will see later that when the initial term z_0 is small, the sequence $z_n = P_c^n(z_0)$, $n = 1, 2, \ldots$, behaves in dramatically different ways for different values of c. However, when the initial term z_0 is large, the sequence behaves in essentially the same way for all values of c, as we now show.

Theorem 2.2

Let $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$. Then, for $|z_0| > r_c$,

$$|P_c^n(z_0)|, \quad n = 0, 1, 2, \ldots,$$

is an increasing sequence, and

$$P_c^n(z_0) \to \infty \text{ as } n \to \infty.$$

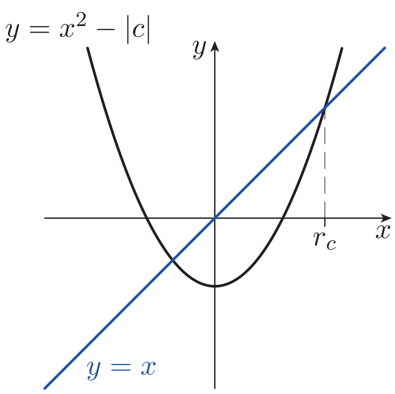


Figure 2.1 The graphs $y = x^2 - |c|$ and $y = x$ intersect at $x = r_c$

Proof First note that, by the backwards form of the Triangle Inequality,

$$|P_c(z)| = |z^2 + c| \geq |z|^2 - |c|. \tag{2.4}$$

The number r_c is the positive solution of the equation $x^2 - |c| = x$ (as shown in Figure 2.1). We claim that if $\varepsilon > 0$, then

$$x^2 - |c| \geq (1 + \varepsilon)x, \quad \text{for } x \geq r_c + \varepsilon. \tag{2.5}$$

Indeed, if $x \geq r_c + \varepsilon$, then

$$\begin{aligned}\frac{x^2 - |c|}{x} &= x - \frac{|c|}{x}\\ &\geq (r_c + \varepsilon) - \frac{|c|}{r_c + \varepsilon}\\ &= \frac{r_c^2 + 2r_c\varepsilon + \varepsilon^2 - |c|}{r_c + \varepsilon}\\ &= \frac{r_c + 2r_c\varepsilon + \varepsilon^2}{r_c + \varepsilon} \quad (\text{since } r_c^2 - |c| = r_c)\\ &\geq \frac{r_c + r_c\varepsilon + \varepsilon + \varepsilon^2}{r_c + \varepsilon} \quad (\text{since } r_c \geq 1)\\ &= 1 + \varepsilon,\end{aligned}$$

as required for inequality (2.5). Inequalities (2.4) and (2.5) now give

$$|P_c(z)| \geq |z|^2 - |c| \geq (1+\varepsilon)|z| > |z|, \quad \text{for } |z| \geq r_c + \varepsilon.$$

If $|z_0| \geq r_c + \varepsilon$, then we can apply this inequality successively to $z_0, P_c(z_0)$, $P_c^2(z_0), \ldots$, to deduce that the sequence $|P_c^n(z_0)|$, $n = 0, 1, 2, \ldots$, is increasing, and

$$|P_c^n(z_0)| \geq (1+\varepsilon)^n |z_0|, \quad \text{for } n = 0, 1, 2, \ldots.$$

Hence $P_c^n(z_0) \to \infty$ as $n \to \infty$.

We have now established the theorem for any initial term z_0 with $|z_0| \geq r_c + \varepsilon$. Since ε is any positive number, we see that the theorem holds for $|z_0| > \varepsilon$. ■

The formula $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$ will play a significant role later in the unit. We indicate in Figure 2.2 the graph of the function $|c| \longmapsto r_c$, together with the graph of the identity function $|c| \longmapsto |c|$ for comparison. Notice, in particular, that

$$|c| \leq r_c \iff |c| \leq 2.$$

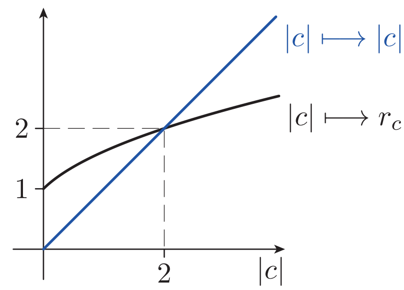


Figure 2.2 The graph of $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$

Remarks

1. If $c = 0$, then $r_c = 1$, so $P_c^n(z_0) \to \infty$ as $n \to \infty$, whenever $|z_0| > 1$, which is the middle line of properties (2.3).
2. Theorem 2.2 can be interpreted as saying that ∞ is an attracting fixed point of the quadratic function P_c. To discover the nature of this fixed point, we can use the conjugating function $h(z) = 1/z$, which moves the fixed point from ∞ to 0. By this means, the iteration sequence

   $$z_{n+1} = z_n^2 + c, \quad n = 0, 1, 2, \ldots,$$

   is conjugate to the iteration sequence (w_n) where

   $$\frac{1}{w_{n+1}} = \left(\frac{1}{w_n}\right)^2 + c, \quad n = 0, 1, 2, \ldots;$$

   that is,

   $$w_{n+1} = \frac{w_n^2}{1 + cw_n^2}, \quad n = 0, 1, 2, \ldots.$$

The corresponding function $Q_c(w) = w^2/(1 + cw^2)$ has a fixed point at 0, which is super-attracting. In this way ∞ can be considered to be a super-attracting fixed point of P_c.

Exercise 2.4

(a) Verify that

$$r_0 = 1, \quad r_i = \tfrac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad r_{-2} = 2.$$

(b) Show that the sequences $(P_0^n(1))$ and $(P_{-2}^n(2))$ are both constant. What do these sequences tell you about the values r_0 and r_{-2} in relation to Theorem 2.2?

We now investigate the set of *all* points that are attracted to ∞, or 'escape' to ∞ under iteration of P_c. We call this set the *escape set*; its complement turns out to be the set of points that we 'keep' under iteration of P_c.

Definitions

For $c \in \mathbb{C}$, the **escape set** of P_c is

$$E_c = \{z : P_c^n(z) \to \infty \text{ as } n \to \infty\}.$$

The complement of E_c is denoted by K_c and is called the **keep set**.

Remarks

1. The escape set E_c can be thought of as the basin of attraction of ∞.
2. The names 'escape set' and 'keep set' are chosen here for convenience. For a general entire function f, the set of points that escape to ∞ under iteration of f is usually called the *escaping set* of f, and it is an object of current research interest. There is no standard name for the complement of the escaping set of an entire function.

For an example of an escape set and a keep set, we know by properties (2.3) that

$$\begin{aligned} E_0 &= \left\{z : P_0^n(z) = z^{2^n} \to \infty \text{ as } n \to \infty\right\} \\ &= \{z : |z| > 1\}, \end{aligned}$$

and hence that

$$\begin{aligned} K_0 &= \mathbb{C} - E_0 \\ &= \{z : |z| \le 1\}. \end{aligned}$$

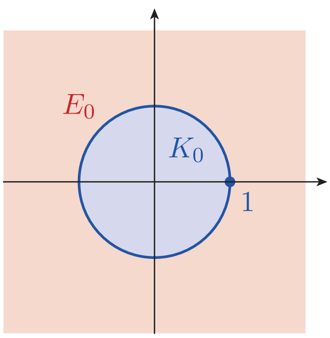


Figure 2.3 The escape set E_0 and keep set K_0

The sets E_0 and K_0 are illustrated in Figure 2.3.

This example is rather misleading since for most values of c, the escape set E_c does not have a simple shape. Figure 2.4 shows several examples of sets E_c and K_c.

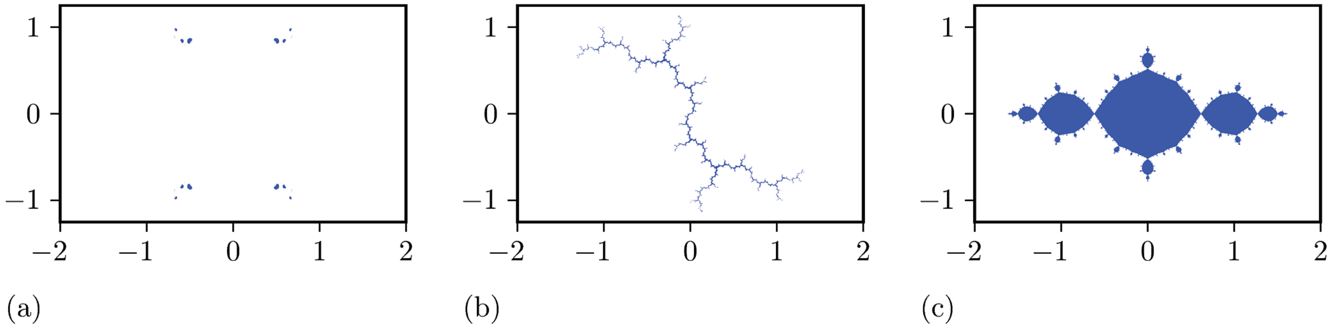


Figure 2.4 The escape set E_c (white) and the keep set K_c (blue) for (a) $c = 1$, (b) $c = i$, (c) $c = -1$

In fact, $c = -2$ is the only value other than $c = 0$ for which E_c and K_c have simple shapes. We now ask you to investigate this case.

Exercise 2.5

Let L be the line segment $\{x + iy : |x| \leq 2,\ y = 0\}$, which is the closed interval $[-2, 2]$ on the real axis.

(a) Let

$$z_{n+1} = z_n^2 - 2, \quad n = 0, 1, 2, \ldots.$$

Prove that

(i) if $z_0 \in L$, then $z_n \in L$, for $n = 1, 2, \ldots$

(ii) if $z_0 \in \mathbb{C} - L$, then $z_n \in \mathbb{C} - L$, for $n = 1, 2, \ldots$.

(b) Prove that if $z_0 \in \mathbb{C} - L$, then the sequence (z_n) in part (a) is conjugate to an iteration sequence of the form

$$w_{n+1} = w_n^2, \quad n = 0, 1, 2, \ldots,$$

and deduce that $z_n \to \infty$ as $n \to \infty$.

(*Hint*: Recall from Theorem 3.1 of Unit D1 that the restriction of the Joukowski function $J(w) = w + 1/w$ to $\{w : |w| > 1\}$ is a one-to-one conformal mapping onto $\mathbb{C} - L$. Put $w_n = J^{-1}(z_n)$, for $n = 0, 1, 2, \ldots$, and verify that $J(w_{n+1}) = J(w_n^2)$.)

(c) Deduce that $E_{-2} = \mathbb{C} - L$ and $K_{-2} = L$ (see Figure 2.5).

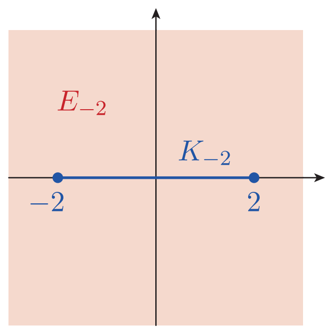


Figure 2.5 The escape set E_{-2} and keep set K_{-2}

Though the set E_c is usually very complicated, a number of general observations can be made about it. For example, we can show that E_c has the property of being *completely invariant* under P_c.

Definition

A set A is **completely invariant** under a function f if

$$z \in A \iff f(z) \in A.$$

For example, it is easy to check that each of the sets $\mathbb{C}$, $\{0\}$, $\mathbb{C} - \{0\}$, $\{z : |z| < 1\}$, $\{z : |z| = 1\}$ and $\{z : |z| > 1\}$ is completely invariant under the function $f(z) = z^2$.

If A is completely invariant under f and $z \in A$, then $f(z)$ and all its iterates also lie in A and, moreover, any point whose iterates eventually lie in A must itself lie in A.

The following result lists several key facts about E_c and K_c.

Theorem 2.3

For each $c \in \mathbb{C}$, the escape set E_c and the keep set K_c have the following properties:

(a) $E_c \supseteq \{z : |z| > r_c\}$ and $K_c \subseteq \{z : |z| \leq r_c\}$

(b) E_c is open and K_c is closed

(c) $E_c \neq \mathbb{C}$ and $K_c \neq \varnothing$

(d) E_c and K_c are each completely invariant under P_c

(e) E_c and K_c are each symmetric under rotation by π about 0

(f) E_c is (pathwise) connected and K_c has no holes in it.

Remarks

1. Each of these properties of K_c is equivalent to the corresponding property of E_c. Thus we need only prove the results for E_c.
2. Properties (a) and (d) tell us that if $z \in K_c$, then $|P_c^n(z)| \leq r_c$, for $n = 1, 2, \ldots$.
3. Properties (a), (b) and (c) tell us that K_c is a non-empty compact set.
4. Property (e) says that $z \in E_c \iff -z \in E_c$ and $z \in K_c \iff -z \in K_c$.
5. Recall from Subsection 4.3 of Unit A3 that a set is said to be (pathwise) connected if each pair of points in the set can be joined by a path lying in the set.
6. The statement in property (f) that K_c has no holes can be expressed formally as 'K_c is simply connected'; this concept was introduced for regions in Subsection 1.1 of Unit B2, but here it applies to the closed set K_c. The proof of property (f) is challenging and may be omitted on a first reading.

Proof

(a) We have already proved part (a) in Theorem 2.2.

(b) Suppose that $z_0 \in E_c$. Then, by definition, $P_c^n(z_0) \to \infty$ as $n \to \infty$. Thus, for some n_0, $|P_c^{n_0}(z_0)| > r_c$.

Let $\varepsilon = |P_c^{n_0}(z_0)| - r_c$. Since $\varepsilon > 0$ and $P_c^{n_0}$ is a polynomial function, $P_c^{n_0}$ is continuous at z_0, so there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |P_c^{n_0}(z) - P_c^{n_0}(z_0)| < \varepsilon,$$

and hence

$$|z - z_0| < \delta \implies |P_c^{n_0}(z)| > r_c$$

(see Figure 2.6). It follows from Theorem 2.2 that

$$|z - z_0| < \delta \implies P_c^{n_0}(z) \in E_c,$$

so

$$|z - z_0| < \delta \implies P_c^n(z) \to \infty \text{ as } n \to \infty.$$

Hence $\{z : |z - z_0| < \delta\} \subseteq E_c$, which shows that E_c is open.

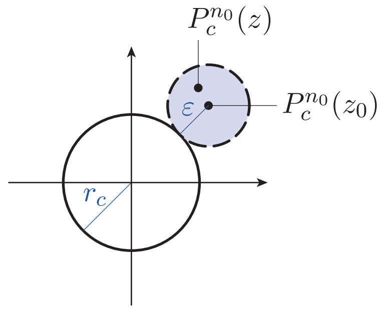


Figure 2.6 The disc $\{z : |z - P_c^{n_0}(z_0)| < \varepsilon\}$ lies outside the circle $\{z : |z| \leq r_c\}$

(c) The set E_c is not the whole of $\mathbb{C}$ because it does not contain the fixed point(s) of P_c, found in Exercise 2.3. (These must lie in K_c.)

(d) We ask you to prove part (d) in Exercise 2.6.

(e) Since

$$P_c^n(-z) = P_c^n(z), \quad \text{for } n = 1, 2, \ldots,$$

because P_c^n is an even function (by Exercise 2.2(b)), we have

$$P_c^n(-z) \to \infty \iff P_c^n(z) \to \infty,$$

so

$$-z \in E_c \iff z \in E_c.$$

(f) Finally, we show that the escape set E_c is (pathwise) connected, that is, each pair of points in E_c can be joined by a path lying entirely in E_c. Since the set $A_c = \{z : |z| > r_c\}$ is connected and $A_c \subseteq E_c$, it is sufficient to show that each point α in E_c can be joined to some point of A_c by a path in E_c. We prove this by contradiction.

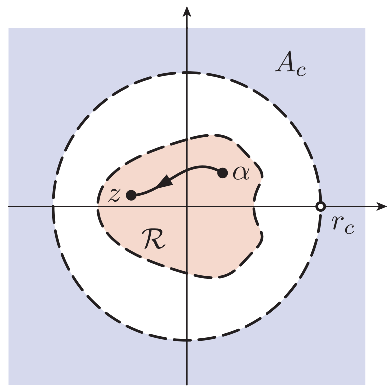


Figure 2.7 The sets $A_c = \{z : |z| > r_c\}$ and $\mathcal{R}$

Suppose in fact that α is a point of E_c that *cannot* be joined to the set A_c by a path in E_c. We define the set

$$\mathcal{R} = \{z \in E_c : z \text{ can be joined to } \alpha \text{ by a path lying entirely in } E_c\},$$

illustrated in Figure 2.7. Then $\mathcal{R} \neq \varnothing$ (because $\alpha \in \mathcal{R}$), $\mathcal{R}$ is open (because if z can be joined to α in E_c, then so can points of any open disc in E_c with centre z, and such a disc exists since E_c is open), and $\mathcal{R}$ is connected (because pairs of points in $\mathcal{R}$ can be joined in $\mathcal{R}$ via α). Thus $\mathcal{R}$ is a region and $\mathcal{R} \subseteq E_c$. Since α cannot be joined in $\mathcal{R}$ to any point of A_c, we deduce that no point in $\mathcal{R}$ can be joined in $\mathcal{R}$ to any point of A_c, since $\mathcal{R}$ is connected. We deduce that $\mathcal{R} \subseteq \{z : |z| \leq r_c\}$, so $\mathcal{R}$ is bounded.

We can now use the Maximum Principle (Theorem 4.2 of Unit C2). If $z \in \partial\mathcal{R}$, then z does not lie in E_c (otherwise, we could enlarge $\mathcal{R}$ slightly, since E_c is open). Thus if $z \in \partial\mathcal{R}$, then

$$|P_c^n(z)| \leq r_c, \quad \text{for } n = 1, 2, \ldots.$$

By applying the Maximum Principle to each of the polynomial functions P_c^n on $\mathcal{R}$, we see that if $z \in \overline{\mathcal{R}}$, then

$$|P_c^n(z)| \leq r_c, \quad \text{for } n = 1, 2, \ldots,$$

which contradicts the fact that $\mathcal{R} \subseteq E_c$. Hence E_c is (pathwise) connected. ■

Exercise 2.6

Prove part (d) of Theorem 2.3.

2.3 Periodic points

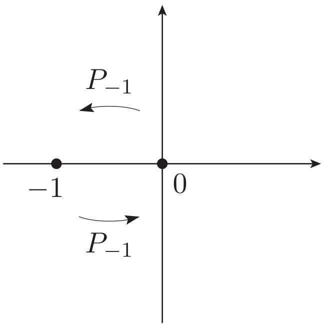


Figure 2.8 P_{-1} interchanges the points 0 and -1

Figure 2.4 shows that the keep sets K_c have remarkably diverse forms. In order to investigate the shape of K_c, we need to identify as many points in K_c as possible. We already know from Exercise 2.3(a) that the fixed points of P_c are $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, and these lie in K_c. We can find other points of K_c by generalising the notion of a fixed point.

As an example, consider the function $P_{-1}(z) = z^2 - 1$, whose two fixed points $\frac{1}{2}(1 \pm \sqrt{5})$ must lie in K_{-1}. In addition, the points 0 and -1 have the property that

$$P_{-1}(0) = -1 \quad \text{and} \quad P_{-1}(-1) = 0 \tag{2.6}$$

(see Figure 2.8). Thus the sequence $(P_{-1}^n(0))$ cycles endlessly between the points 0 and -1, as does the sequence $(P_{-1}^n(-1))$. So 0 and -1 must both lie in K_{-1}. Using the fact that K_{-1} is symmetric under a rotation by π about 0 (by Theorem 2.3(e)), we can begin to build up a picture of K_{-1} (see Figure 2.9). This is very far from the complicated set in Figure 2.4(c), but at least it is a start!

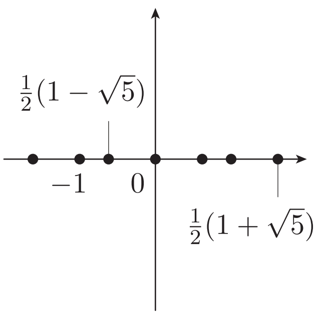


Figure 2.9 Some points in K_{-1}

The fact that 0 and -1 satisfy equations (2.6) can be interpreted as saying that 0 and -1 are both fixed points of the *second* iterate

$$\begin{aligned} P_{-1}^2(z) &= P_{-1}(z^2 - 1) \\ &= (z^2 - 1)^2 - 1 \\ &= z^4 - 2z^2. \end{aligned}$$

Indeed, it is evident that $P_{-1}^2(0) = 0$ and $P_{-1}^2(-1) = -1$.

Points of this type, which are fixed points of some higher iterate of a function f, are called *periodic points* of f: they repeat periodically under iteration of f.

Definitions

The point α is a **periodic point**, with **period** p, of a function f if

$$f^p(\alpha) = \alpha, \text{ but } f^k(\alpha) \neq \alpha, \text{ for } k = 1, 2, \ldots, p-1.$$

The p points

$$\alpha,\ f(\alpha),\ f^2(\alpha),\ \ldots,\ f^{p-1}(\alpha)$$

then form a **cycle of period p**, or a **p-cycle**, of f.

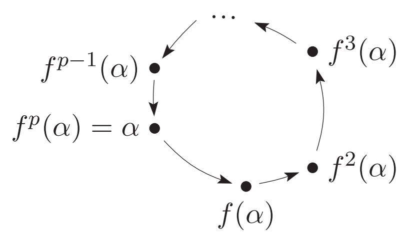


Figure 2.10 A p-cycle of f

A p-cycle of f is illustrated in Figure 2.10.

Remarks

1. The definition states that the period p is the *smallest* positive integer such that $f^p(\alpha) = \alpha$.
2. Note that if we apply f repeatedly to any point of a p-cycle, then we always obtain other points of the p-cycle, each of which is a periodic point with period p. Also, the points of a p-cycle must be distinct. For if we had

   $$f^k(\alpha) = f^\ell(\alpha), \quad \text{where } 0 \leq k < \ell < p,$$

   then it would follow that

   $$\alpha = f^p(\alpha) = f^{p-\ell}(f^\ell(\alpha)) = f^{p-\ell}(f^k(\alpha)) = f^{p-\ell+k}(\alpha),$$

   which is a contradiction, because $0 < p - \ell + k < p$. Therefore the points of a p-cycle are each distinct periodic points of f with period p.
3. A fixed point of f is a 1-cycle of f.
4. Any periodic point α of P_c lies in the keep set K_c, since the sequence $(P_c^n(\alpha))$ takes only finitely many values.

Determining the periodic points with period p of a given function f, for a given $p > 1$, is usually more difficult than determining the fixed points of f. This is because the equation

$$f^p(z) = z \tag{2.7}$$

usually has many more solutions than the equation $f(z) = z$, and the rule for f^p is usually more complicated than the rule for f. Notice, however, that not all solutions of equation (2.7) need to be periodic points with period p. For example, any fixed point of f also satisfies equation (2.7). More generally, if q is a factor of p, then any solution of $f^q(z) = z$ is also a solution of $f^p(z) = z$.

Example 2.1

Determine all periodic points with period 2 of the function $P_0(z) = z^2$, and write down the corresponding 2-cycles.

Solution

Since $P_0^2(z) = (z^2)^2 = z^4$, we have to solve the equation $z^4 = z$:

$$\begin{aligned} z^4 = z &\iff z^4 - z = 0 \\ &\iff z(z^3 - 1) = 0. \end{aligned}$$

The solutions of this quartic equation are 0, 1, $e^{2\pi i/3} = \frac{1}{2}(-1 + i\sqrt{3})$ and $e^{4\pi i/3} = \frac{1}{2}(-1 - i\sqrt{3})$. Of these, the points 0 and 1 are fixed points of P_0, whereas

$$P_0(e^{2\pi i/3}) = (e^{2\pi i/3})^2 = e^{4\pi i/3}$$

and

$$P_0(e^{4\pi i/3}) = (e^{4\pi i/3})^2 = e^{8\pi i/3} = e^{2\pi i/3}.$$

Hence both $e^{2\pi i/3}$ and $e^{4\pi i/3}$ are periodic points of P_0 with period 2, and they belong to the 2-cycle $e^{2\pi i/3}, e^{4\pi i/3}$.

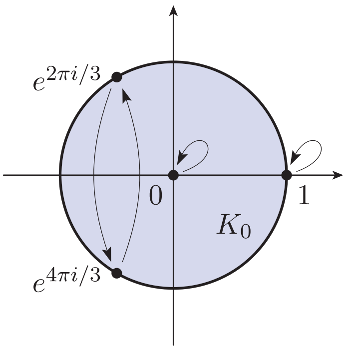


Figure 2.11 A 2-cycle of P_0 and two 1-cycles

Note that, as expected, the 2-cycle found in Example 2.1 lies in the keep set $K_0 = \{z : |z| \leq 1\}$ (see Figure 2.11).

Exercise 2.7

(a) Prove that $-i$ is a periodic point with period 2 of the function $P_i(z) = z^2 + i$. Hence find five points in K_i, none of them fixed points of P_i, and plot them.

(*Hint*: To find three points, use properties of the set K_c from Theorem 2.3. Also, note that the set K_i was plotted in Figure 2.4(b).)

(b) Determine all periodic points with period 3 of the function $P_0(z) = z^2$, and write down the corresponding 3-cycles. Plot all the fixed points, 2-cycles and 3-cycles of P_0 on the same diagram.

(c) Prove that $\frac{1}{2}(-1 + \sqrt{2})$ is a periodic point of $P_{-5/4}$.

We now return to the function $P_{-1}(z) = z^2 - 1$. We saw in equations (2.6) that P_{-1} has the 2-cycle $0, -1$; that is, 0 and -1 are fixed points of the second iterate

$$P_{-1}^2(z) = z^4 - 2z^2,$$

and they are not fixed points of P_{-1}. Since

$$\begin{aligned} (P_{-1}^2)'(z) &= 4z^3 - 4z \\ &= 4z(z^2 - 1), \end{aligned}$$

it follows that

$$(P_{-1}^2)'(0) = 0 \quad \text{and} \quad (P_{-1}^2)'(-1) = 0,$$

so both 0 and -1 are super-attracting fixed points of P_{-1}^2. The fact that these two derivatives have the same value is no accident, as the following theorem shows.

Theorem 2.4

Let $\alpha, f(\alpha), f^2(\alpha), \ldots, f^{p-1}(\alpha)$ form a p-cycle of an analytic function f.

(a) Then the derivative of f^p at α satisfies

$$(f^p)'(\alpha) = f'(\alpha) \times f'(f(\alpha)) \times f'(f^2(\alpha)) \times \cdots \times f'(f^{p-1}(\alpha)),$$

and hence the derivative of f^p takes the same value at each point of the p-cycle; that is,

$$(f^p)'(\alpha) = (f^p)'(f(\alpha)) = (f^p)'(f^2(\alpha)) = \cdots = (f^p)'(f^{p-1}(\alpha)).$$

(b) Let $g = h \circ f \circ h^{-1}$, where h is a one-to-one analytic function, and let $\beta = h(\alpha)$. Then $\beta, g(\beta), g^2(\beta), \ldots, g^{p-1}(\beta)$ is a p-cycle of g, and

$$(g^p)'(\beta) = (f^p)'(\alpha).$$

Proof

(a) Since $f^p(z) = f(f(\cdots(f(z))\cdots))$, where the function f is applied p times, we deduce from repeated applications of the Chain Rule (Theorem 3.1 of Unit A4) that

$$(f^p)'(z) = f'(f^{p-1}(z)) \times \cdots \times f'(f(z)) \times f'(z).$$

Hence

$$(f^p)'(\alpha) = f'(\alpha) \times f'(f(\alpha)) \times \cdots \times f'(f^{p-1}(\alpha)).$$

Thus $(f^p)'(\alpha)$ is the product of the derivatives of f at the points of the p-cycle, so $(f^p)'(f(\alpha))$ is also the product of the derivatives of f at the points of the p-cycle, and similarly for the derivatives of f^p at the other points $f^2(\alpha), \ldots, f^{p-1}(\alpha)$. Hence $(f^p)'$ takes the same value at each point of the p-cycle.

(b) First note that g and f are conjugate functions with conjugating function h, so

$$g^n(w) = h\big(f^n(h^{-1}(w))\big), \quad \text{for } w \in \mathbb{C},\ n = 0, 1, \ldots,$$

since

$$g^n = (h \circ f \circ h^{-1}) \circ (h \circ f \circ h^{-1}) \circ \cdots \circ (h \circ f \circ h^{-1}) = h \circ f^n \circ h^{-1}.$$

Hence $\beta = h(\alpha)$ is a periodic point of the function g of period p, since

$$g^p(\beta) = h\big(f^p(h^{-1}(\beta))\big) = h(f^p(\alpha)) = h(\alpha) = \beta,$$

and $g^k(\beta) \neq \beta$, for $k = 1, 2, \ldots, p-1$.

Since $g^p(w) = h\big(f^p(h^{-1}(w))\big)$, $f^p(\alpha) = \alpha$ and $\alpha = h^{-1}(\beta)$, we deduce by the Chain Rule that

$$\begin{aligned}(g^p)'(\beta) &= h'\big(f^p(h^{-1}(\beta))\big) \times (f^p)'(h^{-1}(\beta)) \times (h^{-1})'(\beta) \\ &= h'(f^p(\alpha)) \times (f^p)'(\alpha) \times (h^{-1})'(\beta) \\ &= h'(\alpha) \times (f^p)'(\alpha) \times (h^{-1})'(\beta).\end{aligned}$$

Now $(h^{-1})'(\beta) = 1/h'(\alpha)$, by the Inverse Function Rule (Theorem 3.2 of Unit A4), so $(g^p)'(\beta) = (f^p)'(\alpha)$, as required. ■

Theorem 2.4(a) allows us to classify the periodic points of an analytic function f by using the number $(f^p)'(\alpha)$, which is called the **multiplier** of the corresponding cycle. We will see shortly that different types of cycle lie in different parts of the keep set.

Definitions

Let α be a periodic point with period p of an analytic function f. Then α and the corresponding p-cycle are

- **attracting** if $\left|(f^p)'(\alpha)\right| < 1$
- **repelling** if $\left|(f^p)'(\alpha)\right| > 1$
- **indifferent** if $\left|(f^p)'(\alpha)\right| = 1$
- **super-attracting** if $(f^p)'(\alpha) = 0$.

Remarks

1. These four types of periodic points generalise the corresponding types of fixed points introduced in Subsection 1.2. Once again, note that super-attracting is a special case of attracting.
2. Theorem 2.4(b) shows that the type of a periodic point is preserved under the operation of conjugacy by a one-to-one analytic function.

In the next example we demonstrate two ways to calculate a multiplier.

Example 2.2

Classify the periodic point $e^{2\pi i/3}$ of $P_0(z) = z^2$.

Solution

In Example 2.1 we found that $e^{2\pi i/3}$ is a periodic point, with period 2, of the function P_0. Since $P_0^2(z) = z^4$ we have $(P_0^2)'(z) = 4z^3$, so the multiplier is

$$(P_0^2)'(e^{2\pi i/3}) = 4(e^{2\pi i/3})^3 = 4.$$

Hence $\left|(P_0^2)'(e^{2\pi i/3})\right| > 1$, so $e^{2\pi i/3}$ is a repelling periodic point of P_0.

Alternatively, the point $e^{2\pi i/3}$ is part of the 2-cycle $e^{2\pi i/3}, e^{4\pi i/3}$ for the function P_0. Thus, by Theorem 2.4(a), the multiplier of the 2-cycle $e^{2\pi i/3}, e^{4\pi i/3}$ is

$$P_0'(e^{2\pi i/3})P_0'(e^{4\pi i/3}) = (2e^{2\pi i/3})(2e^{4\pi i/3}) = 4,$$

since $P_0'(z) = 2z$, as before.

The second method avoids the calculation of the rule for the pth iterate, which can often be complicated.

Exercise 2.8

Classify each of the following periodic points, found in Exercise 2.7.

(a) $-i$, a periodic point of P_i

(b) $e^{2\pi i/7}$, a periodic point of P_0

(c) $\frac{1}{2}(-1+\sqrt{2})$, a periodic point of $P_{-5/4}$

We now look more closely at where the periodic points of P_c lie in K_c. As an example, recall that the function $P_{-1}(z) = z^2 - 1$ has fixed points at $\frac{1}{2}(1 \pm \sqrt{5})$, which are both repelling because $P_{-1}'(z) = 2z$, so

$$\left|P_{-1}'\left(\tfrac{1}{2}(1+\sqrt{5})\right)\right| = 1+\sqrt{5} > 1 \quad \text{and} \quad \left|P_{-1}'\left(\tfrac{1}{2}(1-\sqrt{5})\right)\right| = \sqrt{5}-1 > 1.$$

Also, we saw before Theorem 2.4 that P_{-1} has the super-attracting 2-cycle $0, -1$. In Figure 2.12 these four points are plotted, together with an image of the boundary ∂K_{-1} of K_{-1}.

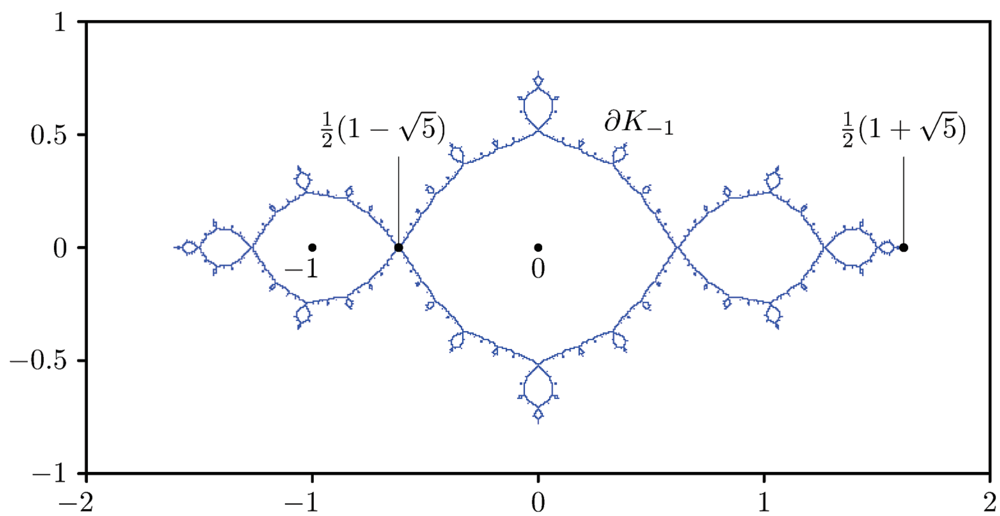


Figure 2.12 The points $\frac{1}{2}(1 \pm \sqrt{5})$ on ∂K_{-1} and the points $0, -1$ in $\operatorname{int} K_{-1}$

This picture suggests that the repelling fixed points $\frac{1}{2}(1 \pm \sqrt{5})$ lie on the boundary of K_{-1}, whereas the super-attracting 2-cycle $0, -1$ lies in the interior $\operatorname{int} K_{-1}$ of K_{-1}.

In fact, the following general result holds. (The definitions of interior point and boundary point are given in Subsection 5.1 of Unit A3.)

Theorem 2.5

Let α be a periodic point of the function P_c.

(a) If α is attracting, then α is an interior point of K_c.

(b) If α is repelling, then α is a boundary point of K_c.

Proof

(a) Suppose that α is an attracting periodic point of P_c with period p. Then α is an attracting fixed point of the pth iterate P_c^p. Hence, by Theorem 1.1, there is an open disc with centre α whose points are attracted to α under iteration of P_c^p. Therefore the points of this open disc do not escape to ∞ under iteration of P_c, so they must all lie in K_c. Hence α is an interior point of K_c.

(b) First suppose that α is a repelling fixed point of P_c, so

$$P_c(\alpha) = \alpha \quad \text{and} \quad |P_c'(\alpha)| > 1.$$

Since $\alpha \in K_c$, we need to show that α is *not* an interior point of K_c, which will imply that $\alpha \in \partial K_c$. But if α were an interior point, then we could choose an open disc $\{z : |z - \alpha| < r\}$ lying in K_c. In that case

$$P_c^n(z) \in K_c, \quad \text{for } |z - \alpha| = \tfrac{1}{2}r \text{ and } n = 1, 2, \ldots,$$

by the complete invariance of K_c, Theorem 2.3(d). Hence

$$|P_c^n(z)| \le r_c = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + |c|}, \quad \text{for } |z - \alpha| = \tfrac{1}{2}r \text{ and } n = 1, 2, \ldots,$$

by Theorem 2.3(a). Now we apply Cauchy's Estimate (Exercise 3.3 of Unit B2) to the first derivative of each of the polynomial functions P_c^n to deduce that

$$\left|(P_c^n)'(\alpha)\right| \le \frac{r_c}{\frac{1}{2}r} = \frac{2r_c}{r}, \quad \text{for } n = 1, 2, \ldots. \tag{2.8}$$

On the other hand, by the Chain Rule,

$$\begin{aligned}(P_c^n)'(\alpha) &= P_c'(P_c^{n-1}(\alpha)) \times \cdots \times P_c'(P_c(\alpha)) \times P_c'(\alpha)\\ &= (P_c'(\alpha))^n,\end{aligned}$$

since, by assumption, α is a fixed point of P_c. Because $|P_c'(\alpha)| > 1$, this implies that the sequence

$$\left|(P_c^n)'(\alpha)\right| = |P_c'(\alpha)|^n, \quad n = 1, 2, \ldots,$$

tends to ∞, contrary to estimate (2.8). Thus α is not an interior point of K_c after all, so $\alpha \in \partial K_c$.

If α is a repelling periodic point of P_c with period $p > 1$, then a similar argument applies (with P_c^p rather than P_c); we omit the details. ■

Remarks

1. Notice in part (a) of Theorem 2.5 that each point of the cycle

$$\alpha,\ P_c(\alpha),\ P_c^2(\alpha),\ \ldots,\ P_c^{p-1}(\alpha),$$

is an attracting periodic point of P_c with period p, by Theorem 2.4(a), so each point of this cycle is an interior point of K_c. The effect of P_c is to map any point z near α to a point $P_c(z)$ near $P_c(\alpha)$, and so on, round and round the cycle; see Figure 2.13.

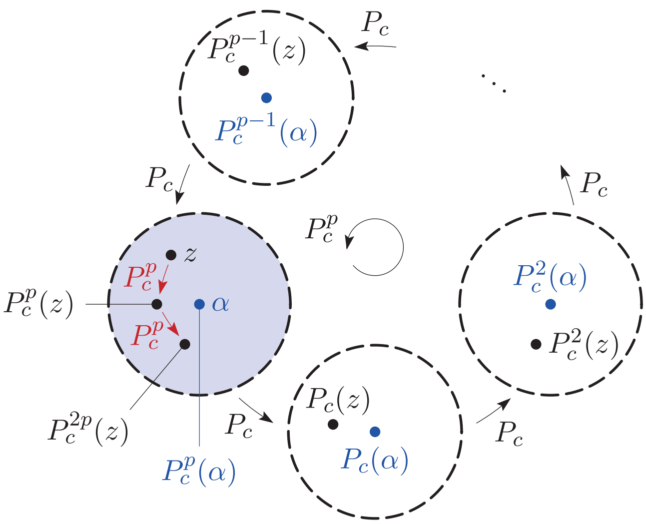


Figure 2.13 Images of a point z under an attracting p-cycle of P_c

Since the sequence $z, P_c^p(z), P_c^{2p}(z), \ldots$ tends to α in the first (shaded) disc, it follows that the sequence

$$P_c(z), P_c^{p+1}(z), P_c^{2p+1}(z), \ldots$$

tends to $P_c(\alpha)$ in the second disc, by the continuity of P_c at α, and similarly in the subsequent discs.

In this way the sequence $(P_c^n(z))$ splits into p convergent subsequences, each converging to a point of the attracting p-cycle. In fact, it can be shown that *every* interior point of K_c is attracted in this way to the same attracting p-cycle; see Figure 2.14, which shows an interior point z_0 of K_{-1} being attracted to the super-attracting 2-cycle $0, -1$.

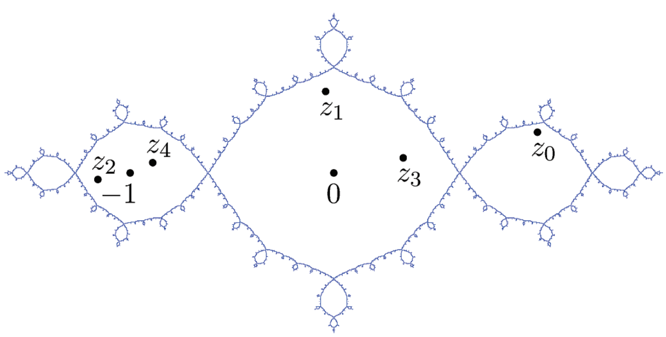


Figure 2.14 The set ∂K_{-1} and the sequence $z_n = P_{-1}^n(z_0),\ n = 0, 1, \ldots$

2. If we know that the set K_c has no interior points for a given value of c, then Theorem 2.5(a) tells us that P_c can have no attracting periodic points. For example, you saw in Exercise 2.5 that

$$K_{-2} = \{x + iy : |x| \leq 2,\ y = 0\},$$

a set which has no interior points, so we deduce that $P_{-2}(z) = z^2 - 2$ has no attracting periodic points, a fact which is not immediately obvious!

3. In view of Theorem 2.5 it is natural to ask where in K_c do any *indifferent* periodic points of P_c lie. The answer is that it depends on the nature of the multiplier of the periodic point in an extremely complicated way: some periodic points lie on ∂K_c and some lie in the interior of K_c.

2.4 The Julia set

This subsection is intended for reading only (it will not be assessed).

We saw in the previous subsection that the boundary of the keep set K_c (which is also the boundary of the escape set E_c) contains all the repelling periodic points of P_c. This boundary is of interest because it forms the 'watershed' between those points that escape to ∞ under iteration of P_c and those which are kept (see Figure 2.15). It is called the *Julia set* of P_c, in honour of the French mathematician Gaston Julia who first studied it in detail.

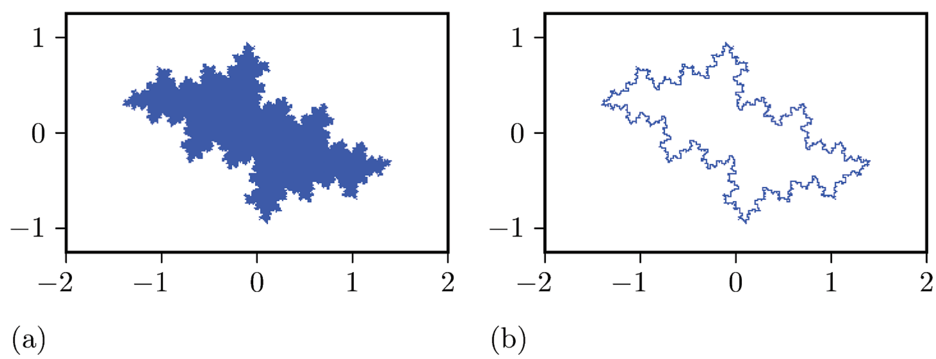


Figure 2.15 For $c = -0.5 + 0.5i$, (a) the keep set K_c and (b) the Julia set J_c

Definition

The **Julia set** J_c of P_c is the boundary of K_c.

Here are two examples of simple Julia sets:

$$K_0 = \{z : |z| \leq 1\}, \quad \text{so } J_0 = \partial K_0 = \{z : |z| = 1\},$$

and

$$K_{-2} = \{x + iy : |x| \leq 2,\ y = 0\}, \quad \text{so } J_{-2} = \partial K_{-2} = K_{-2}.$$

In general, the set K_c is closed and has no holes in it; see Theorem 2.3(b) and (f). Therefore

$$J_c = \partial K_c = K_c - \operatorname{int} K_c,$$

and the keep set K_c can be thought of as J_c together with the 'inside' of J_c. As a result, K_c is often called the *filled Julia set*.

Several general properties of J_c can be deduced from Theorem 2.3. For each $c \in \mathbb{C}$, the Julia set J_c is a non-empty compact subset of $\{z : |z| \leq r_c\}$, which is completely invariant under P_c and symmetric under rotation by π about 0. The complete invariance of J_c holds because K_c is completely invariant (Theorem 2.3(d)), and $\operatorname{int} K_c$ is also completely invariant, which follows from the continuity of P_c and the Open Mapping Theorem (Theorem 3.1 of Unit C2).

The definition of J_c could be used to plot J_c, but there are also other methods for plotting J_c. You saw in Theorem 2.5(b) that J_c contains all the repelling periodic points of P_c, so knowledge of a number of these points gives us some information about the shape of J_c. In fact, it can be shown that

J_c is the smallest closed set that contains all the repelling periodic points of P_c. (2.9)

Thus the shape of J_c is entirely determined by these repelling periodic points. However, calculating a large number of such points would be a difficult task in practice.

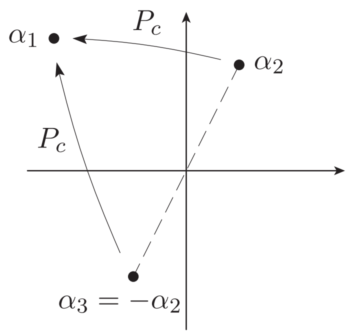


Figure 2.16 P_c maps α_2 and α_3 to α_1

A more satisfactory method of plotting J_c is to use the complete invariance of this set under P_c. The complete invariance tells us that if $\alpha_1 \in J_c$, then the solutions of the equation $P_c(z) = \alpha_1$, that is, $z^2 + c = \alpha_1$, also lie in J_c. This equation has the solutions $\pm\sqrt{\alpha_1 - c}$, which are two new points α_2, α_3 of J_c (see Figure 2.16). (Exceptional cases occur when $\alpha_1 = c$, or when α_1 is a fixed point of P_c. In these cases, there is only one new point.)

Now we can repeat this process with α_2, α_3 in place of α_1 to obtain four new points $\alpha_4, \alpha_5, \alpha_6, \alpha_7$ in J_c. This process, which is illustrated schematically in Figure 2.17, is known as **backward iteration**. It can be shown that

J_c is the smallest closed set that contains all the backward iterates of any point of J_c. (2.10)

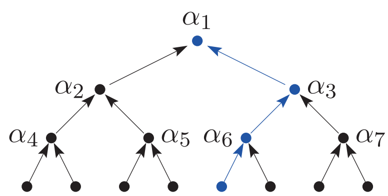


Figure 2.17 Backward iteration of α_1

Thus the shape of J_c is entirely determined by the location of these backward iterates. The calculation of such backward iterates is tricky, even with a computer, because the tree-like structure shown in Figure 2.17 needs careful handling. A common short cut is to make a random choice of square root at each level and plot the resulting sequence; for example,

$$\alpha_1,\ \alpha_3,\ \alpha_6,\ \alpha_{12},\ \ldots.$$

A convenient choice of starting point α_1 for the backward iteration is a repelling or indifferent fixed point of P_c. You can see the result of this method, for several values of c, in Figure 2.18. These particular examples of Julia sets have been given names suggested by their shapes.

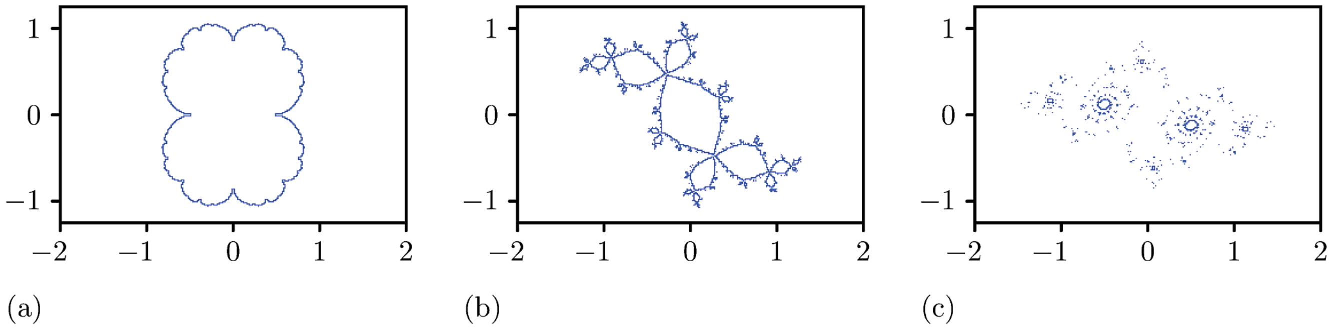


Figure 2.18 The Julia set J_c for (a) $c = 0.25$ (cauliflower), (b) $c = -0.123 + 0.745i$ (rabbit), (c) $c = -0.75 + 0.25i$ (sea horse)

Remarks

1. Property (2.9) implies that near every point of J_c we can find repelling periodic points. The effect of these repelling periodic points is to make the behaviour of the function P_c on J_c extremely unstable, in the sense that points that are close to each other on J_c tend to be pushed apart under iteration of P_c. Such behaviour is often described as *chaotic*.

   By contrast, the behaviour of P_c on $\mathbb{C} - J_c$ is *stable*; that is, points that are close to each other in $\mathbb{C} - J_c$ behave in essentially the same way under iteration of P_c.

   This distinction between stable and unstable behaviour can be used to define the notion of a Julia set for *any* entire function or rational function.

2. Julia sets display a remarkable 'self-similarity' property. For each c, the shape of any part of J_c appears to be repeated all over J_c and is seen even when we zoom in closer and closer to J_c. This is a consequence of the complete invariance of J_c together with property (2.10) of J_c. As a result, Julia sets are often described as *fractals*.

   The name fractal was introduced by the Polish-born mathematician Benoit Mandelbrot (of whom, more later) in 1975 to describe a type of set that is extremely irregular, and yet has an underlying structure that can be seen under magnifications of the set repeated indefinitely. The exact definition of a fractal is not universally agreed, but it has to do with the self-similarity property and with certain methods of measuring the *dimension* of a set that may give non-integer answers! You can learn more about this subject in books and courses on fractal geometry.

Gaston Julia

Gaston Julia (1893–1978) was one of two French mathematicians who developed the theory of iteration of analytic functions in the early part of the twentieth century; the other was Pierre Fatou, who is discussed later in the unit. Julia developed this theory while recovering from serious wounds that he received in World War I (including damage to his face which led him to wear a mask).

In 1917 Julia wrote a major work on iteration which included several breakthroughs, including a key theorem given in Section 4 of this unit, and he was also the first to understand the behaviour of the Newton–Raphson method when applied to cubic equations, as described in Section 1. For this work, Julia won the 1918 Grand Prix des Sciences Mathématiques from the French Académie des Sciences. Julia became a professor at the Sorbonne, and produced important results on many aspects of complex analysis.

Gaston Julia

Further exercises

Exercise 2.9

Show that the iteration sequence

$$z_{n+1} = 3z_n(1 - z_n), \quad n = 0, 1, 2, \ldots,$$

with $z_0 = \frac{1}{2}$, is conjugate to an iteration sequence of the form

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

with $w_0 = 0$, and state the value of d.

Exercise 2.10

(a) Prove that if $|c| \leq \frac{1}{4}$, then

$$|z| \leq \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|} \implies |P_c(z)| \leq \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}.$$

(b) Deduce from part (a) that if $|c| \leq \frac{1}{4}$, then

$$\left\{ z : |z| \leq \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|} \right\} \subseteq K_c.$$

(c) Combine the result of part (b) with Theorem 2.3(a) to show that when c is close to 0, the set K_c is approximately equal to the closed unit disc.

Exercise 2.11

For each of the following functions f and points α, show that α is a periodic point of f, decide whether it is attracting, repelling or indifferent, and identify any attracting fixed points that are super-attracting.

(a) $f(z) = -z, \quad \alpha = i$ (b) $f(z) = z^2 - 2, \quad \alpha = \frac{1}{2}(-1+\sqrt{5})$

(c) $f(z) = z^3 + i, \quad \alpha = 0$ (d) $f(z) = z^3, \quad \alpha = e^{\pi i/13}$

3 Graphical iteration

After working through this section, you should be able to:

- use *graphical iteration* to determine the behaviour of real iteration sequences
- describe properties of the keep sets K_c, for $c \in \mathbb{R}$, that can be obtained by using graphical iteration.

In this section we make some observations about the nature of the keep sets K_c when c is a *real* number. One simple observation is that if c is real, then K_c is symmetric under reflection in the real axis; see, for example, K_{-1} in Figure 2.4(c). This holds because, for $c \in \mathbb{R}$, the functions P_c^n are real polynomial functions, for $n = 1, 2, \ldots$, so $P_c^n(\overline{z}) = \overline{P_c^n(z)}$, for $z \in \mathbb{C}$, which implies that

$$P_c^n(\overline{z}) \to \infty \text{ as } n \to \infty \iff P_c^n(z) \to \infty \text{ as } n \to \infty.$$

Hence $\overline{z} \in K_c$ if and only if $z \in K_c$.

We obtain some more interesting results by using a technique called *graphical iteration*, which applies only to the iteration of *real* functions.

3.1 Defining graphical iteration

If f is a real function, then any iteration sequence of the form $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$, can be represented graphically by using the graphs of $y = f(x)$ and $y = x$ plotted together, as in Figure 3.1. Note that any point in $\mathbb{R}^2$ where $y = f(x)$ meets $y = x$ corresponds to a fixed point of f (for example, the value a in Figure 3.1).

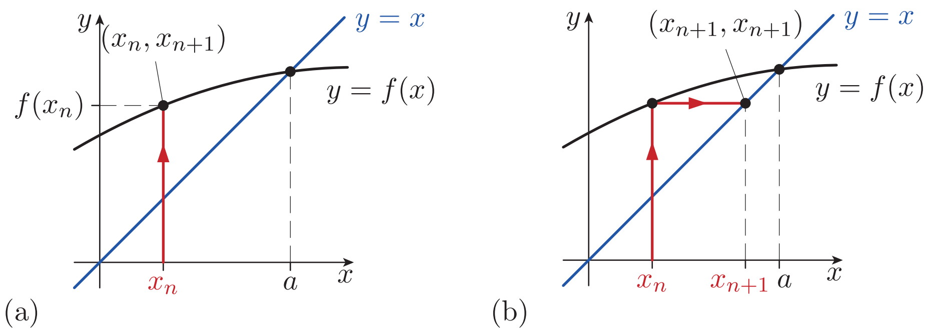


Figure 3.1 Two steps in graphical iteration

Figure 3.1 illustrates a two-step process for finding geometrically the position on the x-axis of the term x_{n+1}, given the position of the term x_n:

(a) draw a vertical line to meet $y = f(x)$ at $(x_n, f(x_n)) = (x_n, x_{n+1})$

(b) draw a horizontal line to meet $y = x$ at (x_{n+1}, x_{n+1}).

Given any initial term x_0, we can apply these two steps repeatedly to construct the sequence (x_n) geometrically, and thus obtain information about the behaviour of (x_n). For example, with the function f in Figure 3.1 and with $x_0 = 0$, we obtain the behaviour illustrated in Figure 3.2, which strongly suggests that (x_n) tends to the fixed point a, where $f(a) = a$.

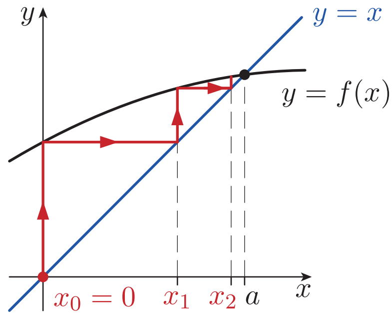


Figure 3.2 Repeating the two-step process with $x_0 = 0$

In the following example we carry out graphical iteration with a particular function f.

Example 3.1

Let $f(x) = \frac{1}{2}x + 1$.

(a) Plot $y = f(x)$ and $y = x$ on the same diagram, and use graphical iteration to plot the iteration sequences

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots,$$

with $x_0 = 0$ and $x_0 = 3$.

(b) Describe the behaviour of each of these sequences (x_n), and check that your answer agrees with the solution to Exercise 1.9(b)(i).

Solution

(a) The graphs of $y = f(x)$ and $y = x$ are plotted in Figure 3.3. They meet at the point $(2, 2)$, which corresponds to the unique fixed point 2 of f. The sequences $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$, with $x_0 = 0$ and $x_0 = 3$, are also plotted.

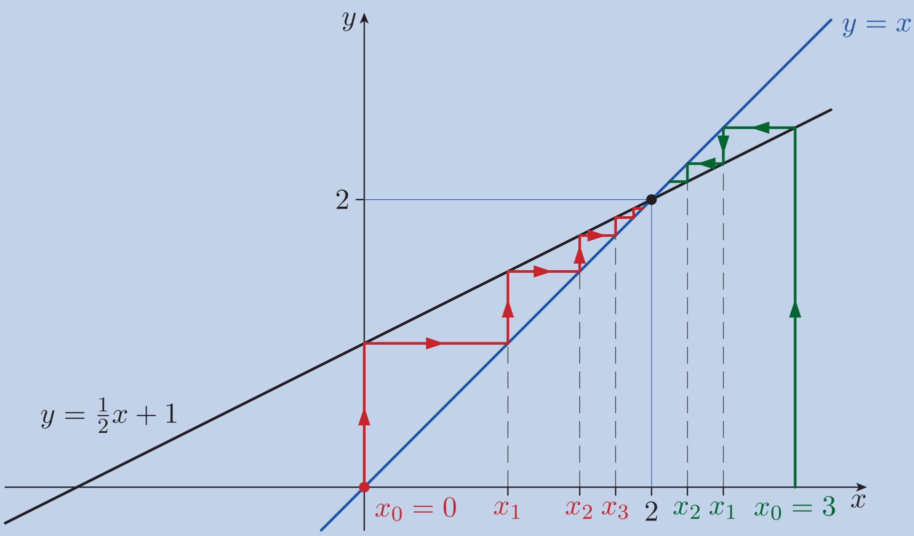


Figure 3.3 Graphical iteration of the function $f(x) = \frac{1}{2}x + 1$

(b) For both values of x_0, Figure 3.3 strongly suggests that $x_n \to 2$ as $n \to \infty$. In Exercise 1.9(b)(i) we found that if $|a| < 1$, then the iteration sequence $z_{n+1} = az_n + b$, $n = 0, 1, 2, \ldots$, converges to (the fixed point) $\alpha = b/(1-a)$ for all z_0. Here we have $a = \frac{1}{2}$, $b = 1$ and $\alpha = 2$, so our answer agrees with this result.

The next exercise gives you a chance to try out graphical iteration.

Exercise 3.1

Let $f(x) = -2x + 1$.

(a) Plot $y = f(x)$ and $y = x$ on the same diagram, and use graphical iteration to plot the iteration sequences

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots,$$

with $x_0 = 0$ and $x_0 = \frac{1}{3}$.

(b) Describe the behaviour of each of these sequences (x_n), and check that your answer agrees with the solution to Exercise 1.9(b)(iii).

3.2 Real quadratic iteration sequences

We now apply graphical iteration to real quadratic iteration sequences of the form

$$x_{n+1} = P_c(x_n) = x_n^2 + c, \quad n = 0, 1, 2, \ldots,$$

where c and x_n are real. A great deal can be said about sequences of this special form, but we confine our attention to a few basic results which throw some light on the corresponding keep sets K_c.

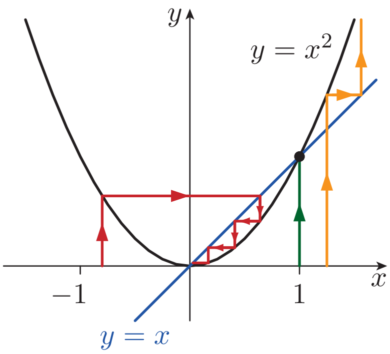


Figure 3.4 Graphical iteration of P_0

To illustrate the method that we use, consider the case $c = 0$. The graphs of $y = P_0(x) = x^2$ and $y = x$ meet at $(0, 0)$ and $(1, 1)$, corresponding to the fixed points 0 and 1 of P_0. The iterations plotted in Figure 3.4 indicate that if $|x| \le 1$, then

$$0 \le P_0^n(x) \le 1, \quad \text{for } n = 1, 2, \ldots.$$

On the other hand, if $|x| > 1$, then

$$P_0^n(x) \to \infty \text{ as } n \to \infty.$$

These results show that the part of the keep set $K_0 = \{z : |z| \le 1\}$ on the real axis is the closed interval $[-1, 1]$, as expected.

From Subsection 2.2, the only values of c for which we know K_c explicitly are $c = 0$ and $c = -2$. For other *real* values of c, we may expect that graphical iteration will give new information about K_c. To see what kind of information can be obtained in this way, try the following exercise.

Exercise 3.2

(a) Plot $y = x^2 + 1$ and $y = x$ on the same diagram, and apply graphical iteration to the sequence $x_{n+1} = x_n^2 + 1$, $n = 0, 1, 2, \dots$, with your own choice of initial term x_0.

(b) Explain why the sequence (x_n) tends to infinity.

(c) What do you deduce about the set K_1?

The solution to Exercise 3.2 suggests that the presence or absence of real fixed points of $P_c(x) = x^2 + c$ is of fundamental importance to the behaviour of iteration sequences of the form $x_{n+1} = x_n^2 + c$. Since the fixed points of P_c are $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$ (see Exercise 2.3(a)), the following lemma is evident.

Lemma 3.1

For $c \in \mathbb{R}$, the function P_c has

(a) no real fixed points if $c > \frac{1}{4}$

(b) a single fixed point $\frac{1}{2}$ if $c = \frac{1}{4}$

(c) two real fixed points $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$ if $c < \frac{1}{4}$.

Part (a) of Lemma 3.1 tells us that if $c > \frac{1}{4}$, then the graph $y = x^2 + c$ lies entirely above $y = x$ (see Figure 3.5). It follows by graphical iteration that if $c > \frac{1}{4}$, then

$$P_c^n(x) \to \infty \text{ as } n \to \infty, \quad \text{for all } x \in \mathbb{R}. \tag{3.1}$$

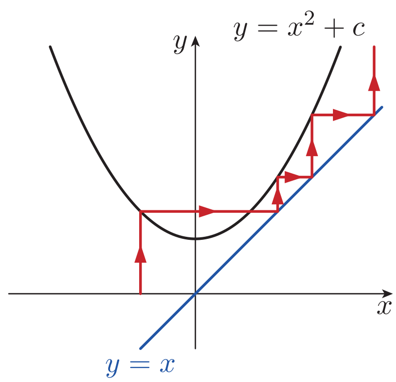


Figure 3.5 Graphical iteration of P_c for $c > \frac{1}{4}$

An algebraic proof of property (3.1) is as follows. Let $c > \frac{1}{4}$. Then we can write $c = \frac{1}{4} + \varepsilon$, where $\varepsilon > 0$. Since $x_{n+1} = x_n^2 + \frac{1}{4} + \varepsilon$, we see that

$$x_{n+1} - x_n = x_n^2 - x_n + \tfrac{1}{4} + \varepsilon = \left(x_n - \tfrac{1}{2}\right)^2 + \varepsilon \geq \varepsilon.$$

Then, writing

$$x_n = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - x_0) + x_0,$$

we deduce that

$$x_n \geq n\varepsilon + x_0, \quad \text{for } n \geq 1,$$

which implies property (3.1).

Thus if $c > \frac{1}{4}$, then each $x \in \mathbb{R}$ escapes to ∞ under iteration of P_c, so no real values of x belong to the keep set K_c. This gives the following result.

Theorem 3.1

If $c > \frac{1}{4}$, then $K_c \cap \mathbb{R} = \varnothing$.

It follows by Theorem 2.3(c) and (e) that if $c > \frac{1}{4}$, then K_c must have points in both the upper and lower half-planes. Hence, for these values of c, the set K_c is in at least two pieces, so it is not (pathwise) connected. Moreover, the plot of K_1 in Figure 2.4(a) suggests that the set K_1 does not meet the imaginary axis. We ask you to prove this property of K_c, for $c > \frac{1}{4}$, in the next exercise.

Exercise 3.3

(a) Show that if c is real and y is real, then $P_c(iy)$ is real.

(b) Deduce from part (a) and Theorem 3.1 that if $c > \frac{1}{4}$, then K_c does not meet the imaginary axis.

If $c \leq \frac{1}{4}$, then, by Lemma 3.1, P_c has either one or two real fixed points, so the keep set K_c does meet the real axis. It turns out that if c lies in the interval $\left[-2, \frac{1}{4}\right]$, then the set $K_c \cap \mathbb{R}$ is equal to the symmetric closed interval

$$I_c = \left[-\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - c},\ \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c}\right].$$

Note that $I_0 = [-1, 1]$ and $I_{-2} = [-2, 2]$, as expected, since $K_0 = \{z : |z| \leq 1\}$ and $K_{-2} = [-2, 2]$.

Theorem 3.2

If $-2 \leq c \leq \frac{1}{4}$, then $K_c \cap \mathbb{R} = I_c$.

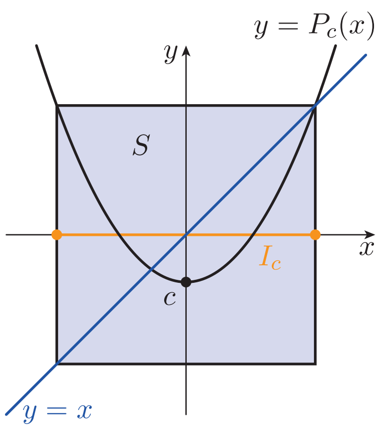


Figure 3.6 Graphs of $y = P_c(x)$ and $y = x$, and the interval I_c

Proof The graphs of $y = P_c(x)$ and $y = x$ are shown in Figure 3.6, together with a filled square S with sides parallel to the axes, which meets the x-axis in the interval I_c. The key to the proof is the observation that if $-2 \leq c \leq \frac{1}{4}$, then the points of the graph of $y = P_c(x)$, for $x \in I_c$, lie in S. If $0 \leq c \leq \frac{1}{4}$, then this is evident because $y = P_c(x)$ does not extend below the x-axis. If $-2 \leq c < 0$, then we need to show that the lowest point of $y = P_c(x)$ does not lie below the bottom edge of the square S; that is,

$$c \geq -\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - c}, \quad \text{for } -2 \leq c < 0. \tag{3.2}$$

To prove this inequality, observe that

$$|c| \leq r_c = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + |c|}, \quad \text{for } 0 \leq |c| \leq 2,$$

which was demonstrated after the proof of Theorem 2.2, and hence

$$-c \leq \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c}, \quad \text{for } -2 \leq c \leq 0,$$

which is equivalent to inequality (3.2).

It follows that

$$\begin{aligned} x \in I_c &\implies P_c^n(x) \in I_c, \quad \text{for } n = 1, 2, \ldots \\ &\implies x \in K_c. \end{aligned}$$

Moreover, graphical iteration shows that

$$\begin{aligned} x \in \mathbb{R} - I_c &\implies P_c^n(x) \to \infty \text{ as } n \to \infty \\ &\implies x \notin K_c. \end{aligned}$$

Thus $K_c \cap \mathbb{R} = I_c$. ■

If $c = -2$, then we know that

$$K_{-2} = [-2, 2] = I_{-2} \subset \mathbb{R}.$$

However, it can be shown that for other values of c in the interval $\left[-2, \frac{1}{4}\right]$, the keep set K_c does not lie entirely on the real axis. For $c < -2$, the keep set K_c has a more interesting intersection with the real axis, which is described in the next result, whose proof may be omitted on a first reading.

Theorem 3.3

If $c < -2$, then the set $K_c \cap \mathbb{R}$ consists of the closed interval I_c with a sequence of disjoint, non-empty, open subintervals of I_c removed. In particular, $0 \notin K_c$.

Proof Let S be the square used in the proof of Theorem 3.2, which meets the x-axis in the interval I_c. Since $c < -2$, we deduce (by graphical iteration) that

$$P_c^n(x) \to \infty \text{ as } n \to \infty, \quad \text{for } x \in \mathbb{R} - I_c, \tag{3.3}$$

so $K_c \cap \mathbb{R} \subseteq I_c$.

Now, since $c < -2$, we can see from the fact that

$$|c| \leq r_c \iff |c| \leq 2$$

(demonstrated after the proof of Theorem 2.2) that the lowest point on $y = P_c(x)$ lies below S, as in Figure 3.7. Therefore the set A_0 of points in I_c that escape from I_c after exactly one iteration of P_c, namely,

$$A_0 = \{x \in I_c : P_c(x) \notin I_c\},$$

is an open subinterval of I_c with centre 0 (see Figure 3.7). In view of property (3.3) it follows that

$$P_c^n(x) \to \infty \text{ as } n \to \infty, \quad \text{for } x \in A_0,$$

so the points of A_0 do not lie in K_c. In particular, $0 \notin K_c$.

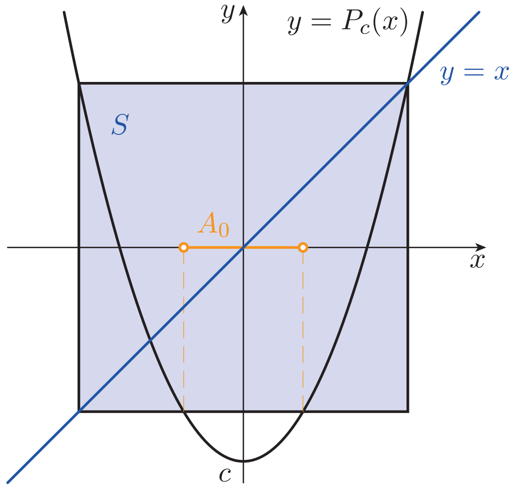


Figure 3.7 Graphs of $y = P_c(x)$ and $y = x$, and the interval A_0

We now consider the set A_1 of points in I_c that remain in I_c for exactly one iteration of P_c, namely,

$$\begin{aligned} A_1 &= \{x \in I_c : P_c(x) \in I_c, \text{ but } P_c^2(x) \notin I_c\} \\ &= \{x \in I_c : P_c(x) \in A_0\}. \end{aligned}$$

The set A_1 (see Figure 3.8) consists of two open subintervals of I_c, which are positioned symmetrically on either side of 0 and which contain the points $\pm\sqrt{-c}$, the zeros of P_c.

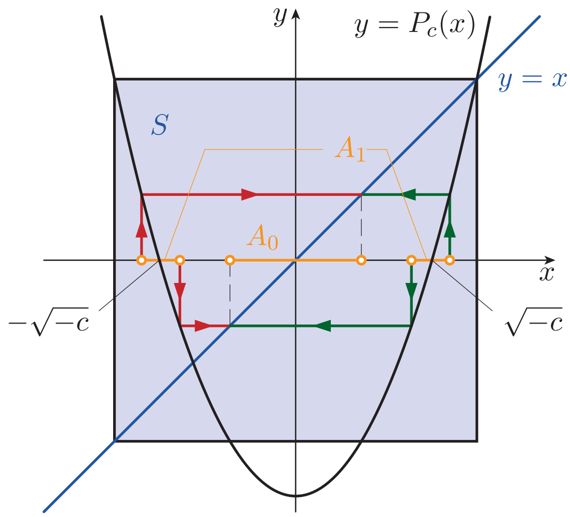


Figure 3.8 The set A_1 consists of two open intervals – the arrowed lines show how the endpoints of these two intervals map to the endpoints of A_0

More generally, we consider the set A_n of points in I_c that remain in I_c for exactly n iterations of P_c, defined inductively as follows:

$$A_n = \{x \in I_c : P_c(x) \in A_{n-1}\}, \quad \text{for } n = 1, 2, \ldots.$$

Since each open interval in A_{n-1} gives rise to two open intervals in A_n, it follows by the Principle of Mathematical Induction that for all $n = 0, 1, 2, \ldots$, the set A_n consists of 2^n disjoint open subintervals of I_c.

Now, any point of I_c that escapes to ∞ under iteration of P_c must lie in exactly one of the sets A_n. Thus the sets A_n are disjoint and

$$K_c \cap \mathbb{R} = I_c - (A_0 \cup A_1 \cup \cdots),$$

which gives the required structure. ■

Remarks

1. The set $K_c \cap \mathbb{R} = I_c - (A_0 \cup A_1 \cup \cdots)$ is infinite because it contains all the endpoints of all the intervals that make up $A_0, A_1, \ldots$. Actually, there are infinitely many points in $K_c \cap \mathbb{R}$ that are *not* endpoints of this type, but these are harder to identify.
2. It can be shown that if $c < -2$, then $K_c \subseteq \mathbb{R}$ (as is the case for $c = -2$).

Exercise 3.4

(a) Show that if c is real and y is real, then $P_c(iy) \leq c$.

(b) Deduce from part (a) and the fact that

$$|c| \leq r_c \iff |c| \leq 2,$$

together with Theorem 2.2, that if $c < -2$, then K_c does not meet the imaginary axis.

Exercise 3.4 shows that if $c < -2$, then the set K_c is in at least two pieces, so (as when $c > \frac{1}{4}$) it is not (pathwise) connected. We pursue the question of the connectedness of K_c in the next section.

Further exercises

Exercise 3.5

Let $P_{1/4}(x) = x^2 + \frac{1}{4}$. Plot $y = P_{1/4}(x)$ and $y = x$ on the same diagram, and use graphical iteration to plot the iteration sequence

$$x_{n+1} = P_{1/4}(x_n), \quad n = 0, 1, 2, \ldots,$$

with $x_0 = 0$. Describe the behaviour of the sequence (x_n).

Exercise 3.6

Use graphical iteration to show that any iteration sequence of the form

$$x_{n+1} = \frac{x_n}{x_n + 1}, \quad n = 0, 1, 2, \ldots,$$

with $x_0 \in \mathbb{R} - A$, where $A = \{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}$, converges to 0.

(*Hint*: Use the fact that if $x = -1/n$, then $x/(x+1) = -1/(n-1)$.)

4 The Mandelbrot set

After working through this section, you should be able to:

- understand the definition of the *Mandelbrot set* M
- use the Fatou–Julia Theorem and its corollaries to determine whether certain points lie in M
- appreciate how a computer can be used to plot M
- show that certain points c lie in M because each of the corresponding functions P_c has an attracting cycle
- appreciate where certain periodic regions of the Mandelbrot set are located by making use of *saddle-node bifurcations* and *period-multiplying bifurcations*.

4.1 Defining the Mandelbrot set

In Section 2 we plotted a number of keep sets K_c for quadratic functions of the form $P_c(z) = z^2 + c$. The shapes of these sets are remarkably varied, but they can be classified into two quite different types: those that are 'all in one piece' (for example, K_0 and K_{-2}) and those that are 'in more than one piece'; see Figure 2.4 for both types. The mathematical name for a set that is 'all in one piece' is *connected*.

In Subsection 4.3 of Unit A3 we introduced the term *pathwise connected* in order to define the key concept of a *region*. A set A is pathwise connected if each pair of points α, β in A can be joined by a path lying entirely in A. It is clear that the sets K_0 and K_{-2} are both pathwise connected, but it is not so evident that the complicated set K_{-1} in Figure 2.4(c) is pathwise connected, even though it does appear to be in one piece.

We now introduce a more general definition of the term *connected*. To avoid confusion, from now on we will write 'pathwise connected' for the concept from Unit A3, not omitting the word 'pathwise'.

Definitions

A set A is **disconnected** if there are disjoint open sets G_1 and G_2 such that

$$A \cap G_1 \neq \varnothing, \quad A \cap G_2 \neq \varnothing \quad \text{and} \quad A \subseteq G_1 \cup G_2.$$

A set A is **connected** if it is not disconnected.

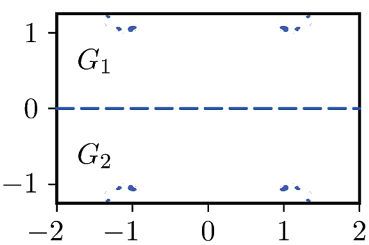


Figure 4.1 $K_{1/2}$ is disconnected

For example, for $c > \frac{1}{4}$ the set K_c is disconnected because, by Theorem 3.1 and Theorem 2.3(c) and (d), it does not meet the real axis, but it does have points in both the upper and lower half-planes. Thus, for $c > \frac{1}{4}$, the definition of disconnected is satisfied for $A = K_c$ with $G_1 = \{z : \operatorname{Im} z > 0\}$ and $G_2 = \{z : \operatorname{Im} z < 0\}$; see Figure 4.1, where $c = \frac{1}{2}$.

Exercise 4.1

Use the fact (proved in Exercise 3.4) that if $c \in \mathbb{R}$ and $c < -2$, then K_c does not meet the imaginary axis, to show that K_c is disconnected for $c < -2$.

It is more difficult to prove that a connected set *is* connected. This is because to do so we need to show that the set is not disconnected, that is, there *do not exist* open sets G_1 and G_2 that disconnect the set. However, it is not too difficult to prove the following result.

Theorem 4.1

Any pathwise connected set is connected.

This result is proved in courses on topology and metric spaces. Note that the converse statement is false, because a connected set need not be pathwise connected.

It follows from Theorem 4.1 that the sets $K_0 = \{z : |z| \leq 1\}$ and $K_{-2} = [-2, 2]$ are connected.

For the moment, we set aside the difficulty of proving that a set is connected and instead concentrate on the set of points c such that K_c is connected; this set is the celebrated *Mandelbrot set*.

Definition

The **Mandelbrot set** is the set M of complex numbers c such that K_c is connected.

Remarks

1. Note that the set M is in the c-plane, often called the *parameter plane*, whereas each keep set K_c lies in the z-plane, often called the *dynamical plane* in this context.
2. The definition of M is often phrased in terms of the connectedness of the Julia set J_c. By using the fact that K_c is simply connected (that is, there are no holes in it), it can be shown that K_c is connected if and only if J_c is connected.

For examples of points in M, we can say that $0 \in M$ and$-2 \in M$ since both K_0 and K_{-2} are connected. On the other hand, by Exercise 4.1 and the discussion preceding it, we can say that $c \notin M$ for all $c > \frac{1}{4}$ and $c < -2$.

The definition of the Mandelbrot set specifies M precisely, but it is difficult to work with. For most values of c we have little idea what K_c looks like, let alone whether or not it is connected! Fortunately, however, there is a numerical criterion for deciding whether K_c is connected, based on the

following fundamental result, the proof of which is outlined in Subsection 4.4.

Theorem 4.2 Fatou–Julia Theorem

For any $c \in \mathbb{C}$,

$$K_c \text{ is connected } \iff 0 \in K_c.$$

Theorem 4.2 states that the keep set K_c is connected if and only if the point 0 does not escape to ∞ under iteration of P_c. For example, 0 lies in both the sets K_0 and K_{-2}, which are connected, but 0 does not lie in K_1, which is disconnected.

To see the power of Theorem 4.2, try the following exercise, which characterises that part of the Mandelbrot set that lies on the real axis.

Exercise 4.2

(a) Use Theorems 3.2 and 4.2 to show that if $c \in \left[-2, \frac{1}{4}\right]$, then $c \in M$.

(b) Deduce that $M \cap \mathbb{R} = \left[-2, \frac{1}{4}\right]$.

Pierre Fatou

Origin of the Fatou–Julia Theorem

Pierre Fatou (1878–1929) was a French astronomer and mathematician. In addition to his fundamental work on complex iteration theory, he proved an important result about the boundary behaviour of analytic functions whose domain is the open unit disc.

Fatou and Julia developed the theory of complex iteration independently and in great detail at about the same time. For example, in 1917–18 they both proved a more general version of Theorem 4.2. After their pioneering work on complex iteration, there were only a few other developments in this field until the explosion of interest in the 1980s, sparked off by the use of computers to plot Julia sets.

The complement of the Julia set is now known as the **Fatou set**.

It is natural to ask why the number 0 appears in this special way in Theorem 4.2. The reason is that the number 0 is the only critical point of each of the functions $P_c(z) = z^2 + c$; that is, it is the only point at which each of the derivatives $P_c'(z) = 2z$ vanishes. By the Local Mapping Theorem (Theorem 3.2 of Unit C2), an analytic function fails to be one-to-one near a critical point, so such points play a significant role in the function's behaviour.

By using Theorem 2.3, we can turn the condition $0 \in K_c$ in Theorem 4.2 into a precise numerical condition that is easier to check.

By the complete invariance of K_c under P_c (see Theorem 2.3(d)),

$$0 \in K_c \implies P_c^n(0) \in K_c, \quad \text{for } n = 0, 1, 2, \ldots.$$

Hence, by Theorem 2.3(a),

$$0 \in K_c \implies |P_c^n(0)| \le r_c, \quad \text{for } n = 0, 1, 2, \ldots, \tag{4.1}$$

where $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$. Since $P_c(0) = c$, the right-hand side of implication (4.1) shows that $|c| \le r_c$, and since $|c| \le r_c \iff |c| \le 2$ (demonstrated after the proof of Theorem 2.2), we see that $r_c \le \frac{1}{2} + \sqrt{\frac{1}{4} + 2} = 2$. Thus

$$0 \in K_c \implies |P_c^n(0)| \le 2, \quad \text{for } n = 1, 2, \ldots. \tag{4.2}$$

On the other hand, if c satisfies the inequalities on the right-hand side of implication (4.2), then the sequence $(P_c^n(0))$ is bounded, so $0 \in K_c$, by the definition of K_c. Hence, by the definition of the set M and Theorem 4.2,

$$\begin{aligned} c \in M &\iff K_c \text{ is connected} \\ &\iff 0 \in K_c \\ &\iff |P_c^n(0)| \le 2, \quad \text{for } n = 1, 2, \ldots. \end{aligned}$$

Thus we obtain the following consequence of Theorem 4.2.

Theorem 4.3

The Mandelbrot set M can be specified as

$$M = \{c : |P_c^n(0)| \le 2, \text{ for } n = 1, 2, \ldots\}.$$

Theorem 4.3 provides a numerical criterion for determining whether or not a point c belongs to M. For any given c, we simply compute the terms of the sequence $(P_c^n(0))$, which are

$$c,\ c^2 + c,\ (c^2 + c)^2 + c,\ \ldots,$$

and try to decide whether *all* these terms lie in $\{z : |z| \le 2\}$. Note that the terms of the sequence $(P_c^n(0))$ are calculated using the recurrence relation

$$P_c^{n+1}(0) = (P_c^n(0))^2 + c, \quad \text{for } n = 1, 2, \ldots.$$

For example, if $c = 0$, then $(P_c^n(0))$ is the constant sequence

$$0, 0, 0, \ldots.$$

Since all these terms lie in $\{z : |z| \le 2\}$, we deduce from Theorem 4.3 that $0 \in M$. On the other hand, if $c = 1$, then the terms of $(P_c^n(0))$ are

$$1, 2, 5, 26, \ldots.$$

These terms do not all lie in $\{z : |z| \le 2\}$, so we deduce from Theorem 4.3 that $1 \notin M$.

Exercise 4.3

Use Theorem 4.3 to determine which of the following points lie in M.

(a) $c = -2$ (b) $c = 1 + i$ (c) $c = i$ (d) $c = i\sqrt{2}$

Theorem 4.3 makes it possible to use a computer to plot an approximation to M. The so-called naive algorithm involves checking the inequality

$$|P_c^n(0)| \leq 2 \tag{4.3}$$

for a large number of points c and for $n = 1, 2, \ldots, N$, where N is a suitably large positive integer. If inequality (4.3) is false for some n, then the corresponding point c lies outside M, but if it is true for $n = 1, 2, \ldots, N$, then c must be in M or be 'close to M'.

Mandelbrot used an algorithm of this kind to plot M, obtaining an image similar in shape to Figure 4.2. The set appears to be very complicated, consisting of many 'blobs' (the main one of which is bounded by a cardioid), which Mandelbrot called *atoms*, all arranged in a highly organised manner. Some of these atoms are stuck together to form a highly complex *molecule*, whereas in Figure 4.2 others appear to float free.

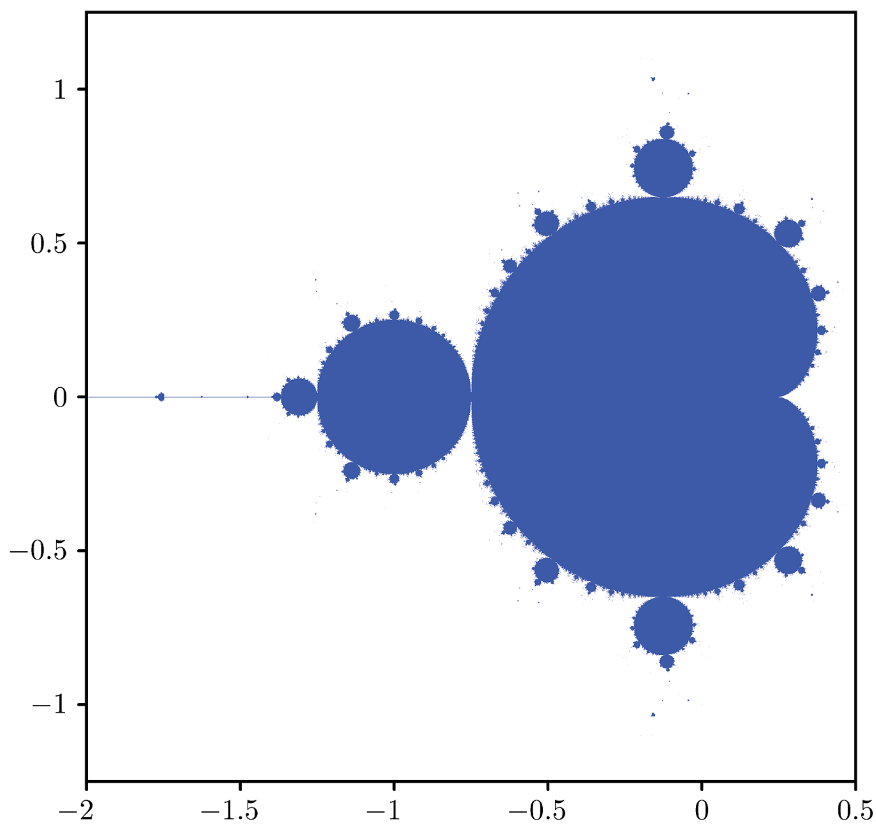


Figure 4.2 The set M in the c-plane plotted using the naive algorithm

History of the Mandelbrot set

The systematic study of the set M began around 1979 when it was plotted by Benoit Mandelbrot (1924–2010). He had previously been plotting Julia sets as part of his pioneering work on fractals.

A very basic plot of the *periodic regions* of M was made at about the same time by Robert Brooks and Peter Matelski; see Figure 4.3. They had encountered the iteration of quadratic functions while studying various groups of Möbius transformations. You will learn about periodic regions in Subsection 4.2.

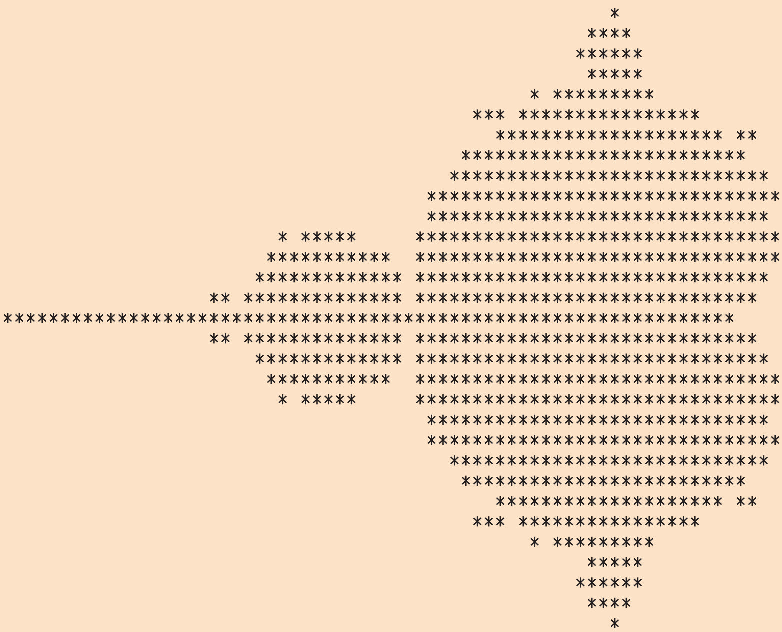


Figure 4.3 Early image of the Mandelbrot set created by Brooks and Matelski

Mandelbrot was born in Warsaw, and his initial training as a mathematician took place at the École Polytechnique, Paris. He started working for IBM in 1958, and became an IBM Fellow in 1974 at the Watson Research Institute in New York State. He contributed to many areas of mathematics and physics, and received many international honours.

Benoit Mandelbrot

We can use Theorem 4.3 to obtain a number of basic properties of M.

Corollary

The Mandelbrot set M

(a) is a compact subset of $\{c : |c| \leq 2\}$

(b) is symmetric under reflection in the real axis

(c) meets the real axis in the interval $\left[-2, \frac{1}{4}\right]$

(d) has no holes in it; that is, $\mathbb{C} - M$ is connected.

Proof First note that each term of the sequence $(P_c^n(0))$ defines a polynomial function of c. Indeed,

$$P_c(0) = c,$$
$$P_c^2(0) = c^2 + c,$$
$$P_c^3(0) = (c^2 + c)^2 + c = c^4 + 2c^3 + c^2 + c,$$

and, in general, $P_c^n(0)$ takes the form

$$P_c^n(0) = c^{2^{n-1}} + 2^{n-2}c^{2^{n-1}-1} + \cdots + c^2 + c, \quad \text{for } n \geq 1.$$

(a) To prove this part we define the sets

$$M_n = \{c : |P_c^n(0)| \leq 2\}, \quad \text{for } n = 1, 2, \ldots,$$

so $M_1 = \{c : |c| \leq 2\}$, $M_2 = \left\{c : \left|c^2 + c\right| \leq 2\right\}$, and so on. Then, by Theorem 4.3,

$$M = M_1 \cap M_2 \cap \cdots,$$

so, in particular, $M \subseteq M_1 = \{c : |c| \leq 2\}$. Thus M is bounded.

Each set M_n is closed because its complement

$$\mathbb{C} - M_n = \{c : |P_c^n(0)| > 2\}$$

is open. Indeed, if $\left|P_{c_0}^n(0)\right| > 2$ for some c_0, then this inequality must hold for all c in some open disc with centre c_0, by the continuity of the function $c \longmapsto |P_c^n(0)|$. It follows that M itself must be closed, because if $c \notin M$, then $c \notin M_n$ for some n, so some open disc with centre c must lie outside M_n and hence outside M, which implies that $\mathbb{C} - M$ is open. Therefore M is closed and bounded; that is, M is compact.

(b) Because $P_c^n(0)$ is a polynomial in c with real coefficients,

$$P_{\overline{c}}^n(0) = \overline{P_c^n(0)}.$$

Hence $|P_{\overline{c}}^n(0)| = |P_c^n(0)|$, for $n = 1, 2, \ldots$, so $\overline{c} \in M$ if and only if $c \in M$, by Theorem 4.3. Hence M is symmetric under reflection in the real axis.

(c) The proof of this part was covered in Exercise 4.2(b).

(d) The proof that $\mathbb{C} - M$ is connected is similar to the proof of part (f) of Theorem 2.3, using the set $\{c : |c| > 2\}$ instead of $\{z : |z| > r_c\}$ and applying the Maximum Principle to the analytic function $c \longmapsto P_c^n(0)$ instead of to $z \longmapsto P_c^n(z)$. We omit the details. ■

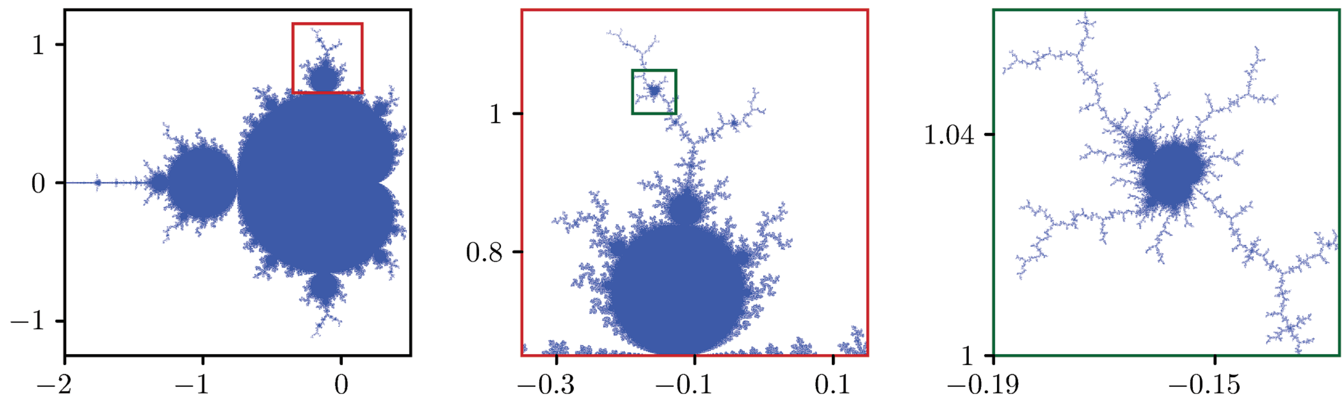
It turns out that the picture of the Mandelbrot set in Figure 4.2 is misleading. Some parts of M are so thin that the naive algorithm fails to detect them. This became clear when the French mathematician Adrien Douady and the American mathematician John Hubbard proved the following remarkable result in 1982. (It was Douady and Hubbard who named the set M after Mandelbrot.)

Theorem 4.4

The Mandelbrot set is connected.

We make no attempt to prove Theorem 4.4, but just remark that the idea of its proof is to construct a conformal mapping from the exterior of M onto the region $\{z : |z| > 1\}$, and the proof uses many results from complex analysis.

Following Theorem 4.4, we can devise more effective algorithms for plotting better renderings of M, such as the one shown in Figure 4.4, with various enlargements corresponding to the small square boxes.



(a) (b) (c)

Figure 4.4 (a) The Mandelbrot set M, and some close-ups of M in (b) and (c)

To gain some insight into why the connectedness of M is so remarkable, we note that the sets

$$M_n = \{c : |P_c^n(0)| \le 2\}, \quad n = 1, 2, \ldots,$$

defined in the proof of the corollary to Theorem 4.3, are nested; that is,

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots.$$

Since $M = M_1 \cap M_2 \cap \cdots$, we can think of M as the limit of this sequence of nested sets.

The nested sets M_n are represented in Figure 4.5, where, for the first few values of n, points of $M_n - M_{n+1}$ are plotted in blue if n is odd and in white if n is even.

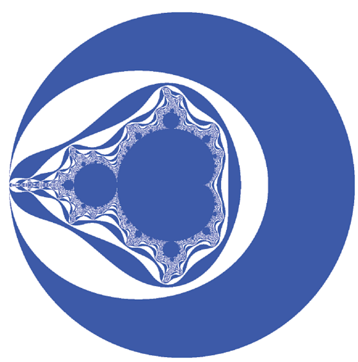


Figure 4.5 The sets M_n

Since $c \in M_n - M_{n+1}$ if and only if

$$|P_c^k(0)| \le 2, \quad \text{for } k = 1, 2, \ldots, n, \text{ but } \left|P_c^{n+1}(0)\right| > 2,$$

these bands indicate how long the corresponding sequences $(P_c^n(0))$ remain in $\{z : |z| \le 2\}$.

It can be shown that the connectedness of M is equivalent to the fact that each of the boundaries $\partial M_n = \{c : |P_c^n(0)| = 2\}$, for $n = 1, 2, \ldots$, forms a simple-closed path; that is, it does not break up into several pieces. This fact can be checked directly for the special cases $\partial M_1 = \{c : |c| = 2\}$ and $\partial M_2 = \{c : |c^2 + c| = 2\}$, but in the case of general n it is far from obvious.

4.2 Inside the Mandelbrot set

Theorem 4.3 provides a good means of showing that a given point c is *not* in the set M. This result is not so helpful, however, as a way to show that c is in M. For some values of c, such as $c = 0$ or $c = -2$, it is possible to check directly that $|P_c^n(0)| \le 2$, for $n = 1, 2, \ldots$, but this is usually not the case. Instead, the following general result can often be used.

Theorem 4.5

If the function P_c has an attracting cycle, then $c \in M$.

Remark

One way to prove Theorem 4.5 is to show that if P_c has an attracting cycle, then the sequence $(P_c^n(0))$ is attracted to this cycle. Since $(P_c^n(0))$ can be attracted to at most one cycle, it follows that P_c has at most one attracting cycle for each value of c. We outline a different proof of Theorem 4.5 in Subsection 4.4.

Theorem 4.5 allows us to identify various key parts of the set M by finding values of c for which the function P_c has an attracting p-cycle for various values of p. There is a particularly elegant method for finding attracting fixed points and attracting 2-cycles of P_c, as we now show.

Theorem 4.6

(a) The function P_c has an attracting fixed point if and only if c satisfies

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c < 3. \tag{4.4}$$

(b) The function P_c has an attracting 2-cycle if and only if c satisfies

$$|c + 1| < \tfrac{1}{4}. \tag{4.5}$$

The proof of Theorem 4.6(a) follows shortly. As you will see there, inequality (4.4) is equivalent to the statement that c lies inside the cardioid with parametrisation

$$\gamma(t) = \tfrac{1}{2}e^{it} - \tfrac{1}{4}e^{2it} \quad (t \in [-\pi, \pi]).$$

This cardioid is the boundary of the main 'atom' of M. Inequality (4.5) says that the point c is in the open disc $\left\{c : |c+1| < \tfrac{1}{4}\right\}$, which lies immediately to the left of the cardioid; see Figure 4.6.

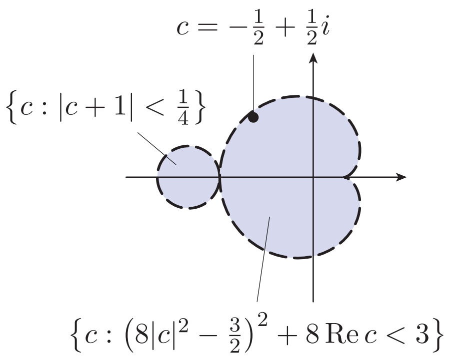


Figure 4.6 The sets of values c where P_c has an attracting fixed point or an attracting 2-cycle

For example, we can check that the point $c = -\frac{1}{2} + \frac{1}{2}i$, shown in Figure 4.6, lies in M by noting that it seems to lie just inside the cardioid. For this value of c, we have $|c|^2 = \frac{1}{2}$ and $\mathrm{Re}\,c = -\frac{1}{2}$, so

$$\begin{aligned}\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c &= \left(4 - \tfrac{3}{2}\right)^2 - 8 \times \tfrac{1}{2}\\ &= \tfrac{9}{4} < 3.\end{aligned}$$

Hence P_c has an attracting fixed point, by Theorem 4.6(a), so $c = -\frac{1}{2} + \frac{1}{2}i$ is in M, by Theorem 4.5.

Exercise 4.4

Prove that each of the following points lies in M.

(a) $c = -0.9 + 0.1i$ (b) $c = 0.2 + 0.5i$

Proof of Theorem 4.6(a) First note that α is a fixed point of P_c if and only if

$$P_c(\alpha) = \alpha^2 + c = \alpha,$$

that is, if and only if

$$c = \alpha - \alpha^2.$$

Moreover, this fixed point is attracting if and only if

$$|P_c'(\alpha)| = |2\alpha| < 1.$$

Thus P_c has an attracting fixed point if and only if c is of the form $\alpha - \alpha^2$, where $|\alpha| < \frac{1}{2}$, that is, if and only if c lies in the image of the open disc $\{z : |z| < \frac{1}{2}\}$ under the function $f(z) = z - z^2$. To understand the nature of this image, we use the approach of Subsection 4.3 of Unit C3, expressing f as a composition of one-to-one conformal mappings. We can write

$$\begin{aligned} z - z^2 &= \tfrac{1}{4} - \left(\tfrac{1}{4} - z + z^2\right) \\ &= \tfrac{1}{4} - \left(\tfrac{1}{2} - z\right)^2, \end{aligned}$$

so f has the effect indicated in Figure 4.7.

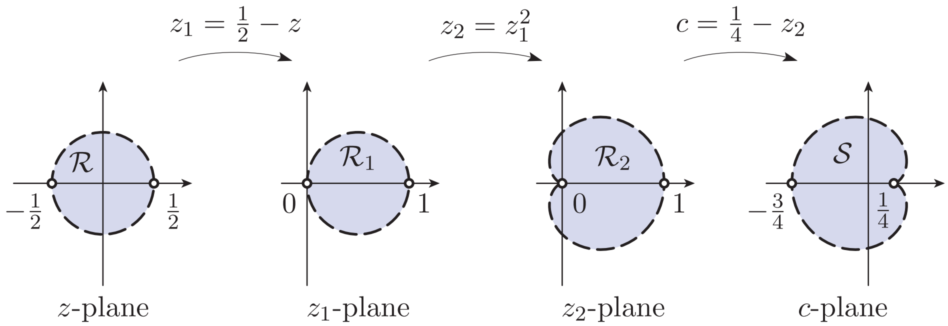


Figure 4.7 Decomposing the function $f(z) = z - z^2$

Note that the cusp of the cardioid arises because the function f satisfies $f'(\frac{1}{2}) = 0$ but $f''(\frac{1}{2}) = -2 \neq 0$. In particular, this function doubles angles between smooth paths emerging from $\frac{1}{2}$.

To complete the proof, we need to show that c lies in the region $\mathcal{S}$ bounded by the cardioid if and only if

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c < 3.$$

This is a rather fiddly technical exercise, which can be done in various ways. One approach is to note that the cardioid itself is the image of the circle $|z| = \frac{1}{2}$ under the function $f(z) = z - z^2$; that is, it is the path in the c-plane with parametrisation

$$\begin{aligned} \gamma(t) &= \tfrac{1}{2}e^{it} - \left(\tfrac{1}{2}e^{it}\right)^2 \\ &= \tfrac{1}{2}(\cos t + i \sin t) - \tfrac{1}{4}(\cos 2t + i \sin 2t) \quad (t \in [-\pi, \pi]). \end{aligned}$$

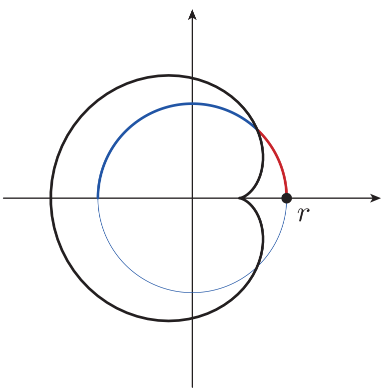


Figure 4.8 A circle $\{z : |z| = r\}$ and the cardioid

In Exercise 2.9 of Unit A2 we found that this path has equation

$$4|c|^4 - \tfrac{3}{2}|c|^2 + \tfrac{1}{2}\,\mathrm{Re}\,c = \tfrac{3}{64},$$

which can be rearranged to give

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c = 3.$$

It can then be shown that for $\frac{1}{4} < r < \frac{3}{4}$, the cardioid splits the part of the circle $|c| = r$ in the upper half-plane into two arcs (see Figure 4.8).

On the left (blue) arc, which is inside the cardioid, inequality (4.4) holds (because $|c| = r$ is a fixed value but $\mathrm{Re}\,c$ decreases), whereas on the right (red) arc we have the opposite inequality. For other positive values of r, the cardioid does not meet this semicircle, and the direction of the inequality can be found by considering its nature at $c = \pm r$. We omit the details. ■

To prove Theorem 4.6(b) we need the following useful lemma.

Lemma 4.1

Suppose that $c \neq -\frac{3}{4}$. Then P_c has a single 2-cycle α_1, α_2, where

$$\alpha_1 = -\tfrac{1}{2} + \sqrt{-\tfrac{3}{4} - c}, \quad \alpha_2 = -\tfrac{1}{2} - \sqrt{-\tfrac{3}{4} - c},$$

with multiplier

$$(P_c^2)'(\alpha_1) = 4\alpha_1\alpha_2 = 4(c+1).$$

If $c = -\frac{3}{4}$, then $\alpha_1 = \alpha_2 = -\frac{1}{2}$, which is a fixed point of $P_{-3/4}$, not a 2-cycle ($P_{-3/4}$ does not have any 2-cycles).

Proof Since $P_c^2(z) = (z^2+c)^2 + c = (P_c(z))^2 + c$, we have

$$\begin{aligned} P_c^2(z) - z &= (P_c(z))^2 + c - z \\ &= (P_c(z))^2 - z^2 + P_c(z) - z \\ &= (P_c(z) - z)(P_c(z) + z) + P_c(z) - z \\ &= (P_c(z) - z)(P_c(z) + z + 1) \\ &= (P_c(z) - z)(z^2 + z + c + 1). \end{aligned} \tag{4.6}$$

The 2-cycles of P_c are those solutions of $P_c^2(z) - z = 0$ that are not solutions of $P_c(z) - z = 0$. Hence, by equation (4.6), they are the solutions of

$$z^2 + z + c + 1 = 0.$$

So P_c has a single 2-cycle α_1, α_2, where

$$\alpha_1 = -\tfrac{1}{2} + \sqrt{-\tfrac{3}{4} - c}, \quad \alpha_2 = -\tfrac{1}{2} - \sqrt{-\tfrac{3}{4} - c}.$$

By Theorem 2.4(a), the multiplier of the 2-cycle α_1, α_2 is

$$\begin{aligned} (P_c^2)'(\alpha_1) &= P_c'(\alpha_1)P_c'(\alpha_2) \\ &= (2\alpha_1)(2\alpha_2) \\ &= 4\alpha_1\alpha_2 \\ &= 4(c+1). \end{aligned}$$

■

Exercise 4.5

Use Lemma 4.1 to prove that P_c has an attracting 2-cycle if and only if $|c+1| < \frac{1}{4}$, thus proving Theorem 4.6(b).

Numerical experiments suggest that the Mandelbrot set contains many simply connected regions, each with an associated period p such that for every c in the interior of the region, the function P_c has an attracting p-cycle. This motivates the following definition.

Definition

A **periodic region** is a maximal region $\mathcal{R}$ such that, for some positive integer p,

$$\text{the function } P_c \text{ has an attracting } p\text{-cycle, for all } c \in \mathcal{R}. \tag{4.7}$$

Here the word 'maximal' means that there is no larger region containing $\mathcal{R}$ that satisfies (4.7).

For example, the inside of the cardioid and the open disc specified by inequalities (4.4) and (4.5) are both periodic regions. Unfortunately, no other periodic region in M seems to have such a straightforward characterisation, but we can obtain some information about their location in M by using the following result.

Theorem 4.7

The function P_c has a super-attracting p-cycle if and only if

$$P_c^p(0) = 0, \quad \text{but } P_c^k(0) \neq 0, \text{ for } k = 1, 2, \ldots, p-1. \tag{4.8}$$

Proof By Theorem 2.4(a), any p-cycle

$$\alpha, P_c(\alpha), \ldots, P_c^{p-1}(\alpha)$$

of P_c has multiplier

$$P_c'(\alpha)P_c'(P_c(\alpha)) \cdots P_c'(P_c^{p-1}(\alpha)) = (2\alpha)(2P_c(\alpha)) \cdots (2P_c^{p-1}(\alpha)),$$

since $P_c'(z) = 2z$. Therefore such a p-cycle is super-attracting if and only if one of the points of the p-cycle is 0. But P_c has a p-cycle including 0 if and only if condition (4.8) holds, so the proof is complete. ■

For example, if $p = 1$, then condition (4.8) becomes

$$P_c(0) = c = 0,$$

as expected, since $P_0(z) = z^2$ has the super-attracting fixed point 0.

If $p = 2$, then condition (4.8) becomes

$$P_c^2(0) = c^2 + c = 0, \quad \text{but } P_c(0) = c \neq 0.$$

The only solution is $c = -1$, as expected, since $P_{-1}(z) = z^2 - 1$ has the super-attracting 2-cycle $0, -1$, as we saw just before Theorem 2.4.

Exercise 4.6

Use Theorem 4.7 to show that the function P_c has a super-attracting 3-cycle for precisely three different points c, one of which lies in the interval $[-1.8, -1.7]$ and the other two of which form a pair of complex conjugates.

It can be proved that if P_c has an attracting p-cycle, then c lies in a periodic region that contains exactly one point c_0, say, with a super-attracting p-cycle. We call c_0 the **centre** of the associated periodic region. Figure 4.9 shows the approximate location in the Mandelbrot set of all points c_0 for which P_{c_0} has a super-attracting p-cycle for $p = 1, 2, 3, 4$.

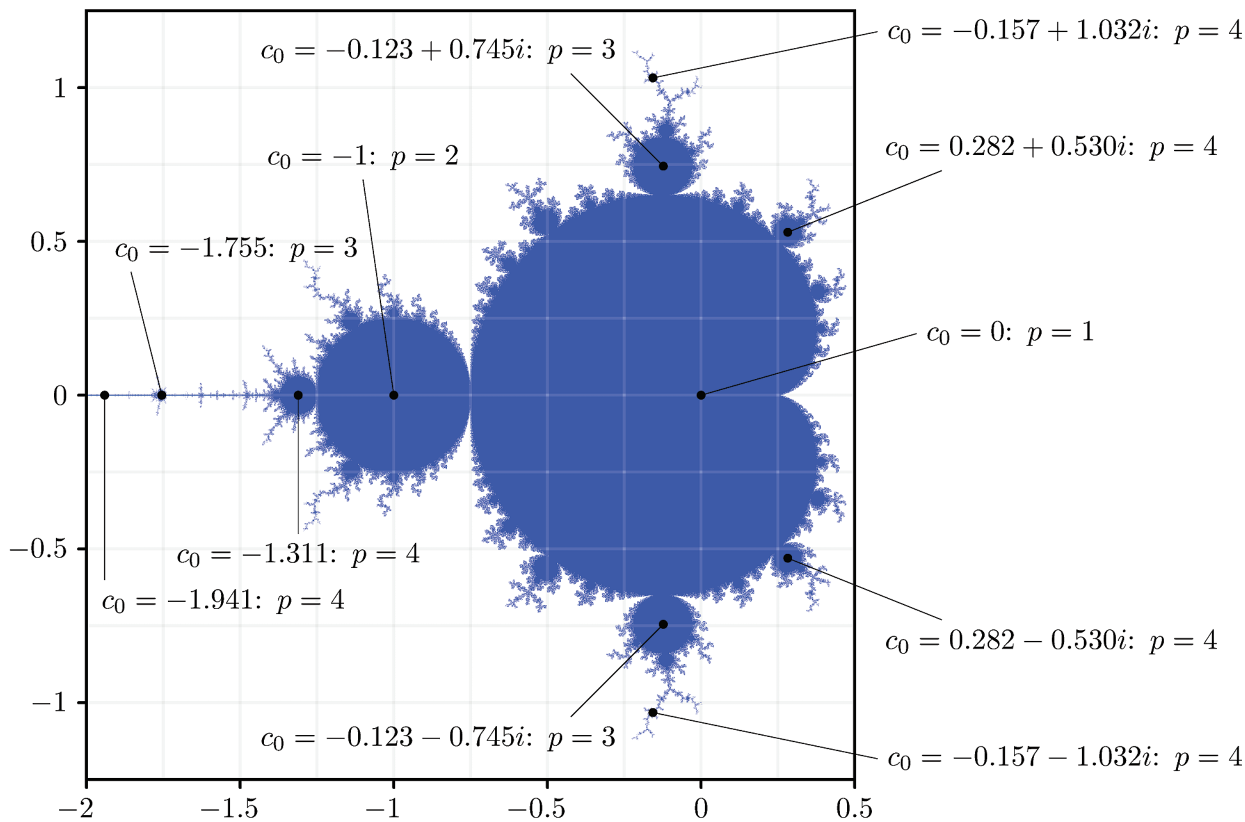


Figure 4.9 The set M showing points c_0 (correct to 3 d.p.) for which P_{c_0} has a super-attracting cycle

In fact, the boundaries of all the periodic regions are either (roughly) circular in shape or shaped like a cardioid, but they are connected together in a complicated manner. We cannot hope to give a full explanation of the way in which the periodic regions of M fit together, though we can gain some insight into their structure by looking more closely at the two sets given by Theorem 4.6. These are plotted in Figure 4.10.

The key points to consider here are $c = \frac{1}{4}$, which is the cusp of the cardioid, and $c = -\frac{3}{4}$, where the cardioid and the circle meet. These are the points where the cycle structure of P_c changes; for example, as c passes

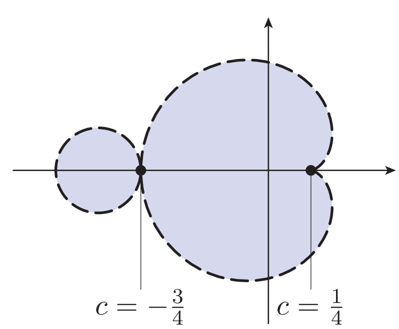


Figure 4.10 Two bifurcation points

through $-\frac{3}{4}$ from the cardioid into the disc, the attracting fixed point of P_c becomes repelling, and the repelling 2-cycle of P_c becomes attracting. At such points, we say that a *bifurcation* occurs. In order to characterise such bifurcations, we look more closely at the cycle structure of P_c for $c = \frac{1}{4}$ and $c = -\frac{3}{4}$.

For $c = \frac{1}{4}$ the function $P_c = P_{1/4}$ has just one fixed point, $\alpha = \frac{1}{2}$, with multiplier

$$P'_c(\alpha) = P'_{1/4}\left(\tfrac{1}{2}\right) = 2\left(\tfrac{1}{2}\right) = 1.$$

For $c = -\frac{3}{4}$ the function $P_c = P_{-3/4}$ has two fixed points, one of which is at $\alpha = -\frac{1}{2}$ with multiplier

$$P'_c(\alpha) = P'_{-3/4}\left(-\tfrac{1}{2}\right) = 2\left(-\tfrac{1}{2}\right) = -1.$$

Thus for both these key values of c, the function P_c has a fixed point α whose multiplier $P'_c(\alpha)$ is a root of unity.

In order to state the next result, we say that the number λ is a **primitive *n*th root of unity** if λ is a root of unity and if n is the smallest positive integer for which $\lambda^n = 1$. For example, -1 is a primitive square root of unity. It is also a fourth root of unity because $(-1)^4 = 1$, but it is not a *primitive* fourth root of unity.

The following theorem, whose proof we omit, gives the two types of bifurcation that can occur when the multiplier of a cycle is a root of unity.

Theorem 4.8

Suppose that the function P_{c_0}, where $c_0 \in \mathbb{C}$, has a p-cycle whose multiplier λ is a root of unity.

(a) **Saddle-node bifurcation at c_0** If $\lambda = 1$, then c_0 is the cusp of a cardioid-shaped periodic region $\mathcal{R}$, such that

$$P_c \text{ has an attracting } p\text{-cycle, for } c \in \mathcal{R}.$$

(b) **Period-multiplying bifurcation at c_0** If λ is a primitive nth root of unity, for $n > 1$, then there are two periodic regions $\mathcal{R}_1$ and $\mathcal{R}_2$ whose boundaries meet at c_0 such that

$$P_c \text{ has an attracting } \begin{cases} p\text{-cycle}, & \text{for } c \in \mathcal{R}_1, \\ np\text{-cycle}, & \text{for } c \in \mathcal{R}_2. \end{cases}$$

Remarks

1. The name *saddle-node* bifurcation arises from the shape of the graph when such a bifurcation occurs within a family of *real* functions.
2. If $n = 2$, then a period-multiplying bifurcation is called a *period-doubling bifurcation*.

In the next two (challenging) exercises we ask you to investigate specific examples of these two types of bifurcations.

Exercise 4.7

(a) Use the solution to Exercise 2.2(a) to determine a polynomial $Q_c(z)$ such that

$$P_c^3(z) - z = (P_c(z) - z)Q_c(z).$$

(b) Verify that

$$P_{-7/4}^3(z) - z = (P_{-7/4}(z) - z)\big(z^3 + \tfrac{1}{2}z^2 - \tfrac{9}{4}z - \tfrac{1}{8}\big)^2.$$

(c) Show that the equation $z^3 + \frac{1}{2}z^2 - \frac{9}{4}z - \frac{1}{8} = 0$ has three real solutions, and that these form a 3-cycle of $P_{-7/4}$.

(d) Deduce that a saddle-node bifurcation occurs at $c = -\frac{7}{4}$, and relate this fact to the solution of Exercise 4.6 and to Figure 4.9.

Exercise 4.8

Show that if $c = \zeta - \zeta^2$, where 2ζ is a root of unity ($\neq 1$), then a period-multiplying bifurcation occurs at c. Relate this fact to Figure 4.9 with $\zeta = -\frac{1}{2}$, $\frac{1}{2}e^{2\pi i/3}$ and $\frac{1}{2}i$.

(*Hint*: First check that ζ is a fixed point of P_c.)

4.3 The structure of the Mandelbrot set

This subsection is intended for reading only (it will not be assessed).

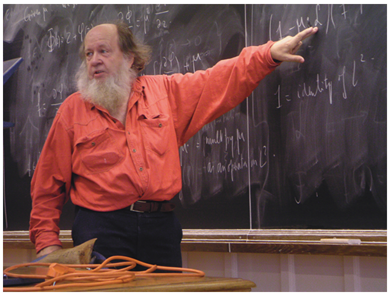
The Mandelbrot set is an incredibly complicated object and there are some major questions about its structure that are unresolved. In this subsection we describe more of what is known about this remarkable set.

Using the approach of Exercise 4.8, we find that each part of the main cardioid in Figure 4.9 is decorated by periodic regions. In a similar way, all these periodic regions are themselves decorated everywhere by further periodic regions, and so on.

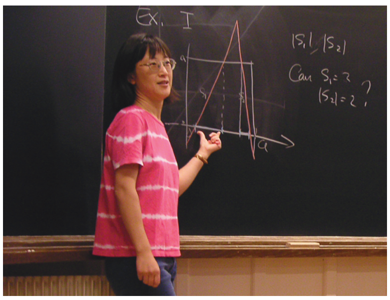
In addition, we find throughout the boundary of M the appearance of small cardioid-shaped periodic regions, arising from saddle-node bifurcations, such as the one at $c = -\frac{7}{4}$ in Exercise 4.7. These cardioid-shaped regions are themselves decorated with smaller periodic regions, as a result of period-multiplying bifurcations, so they give rise to small copies of the Mandelbrot set within itself (see Figure 4.4 and the front cover of this book).

All these periodic regions appear to be connected together in M by a complicated network of 'veins' – recall that the Mandelbrot set *is* connected (Theorem 4.4), as shown by Douady and Hubbard.

The simplest such vein lies along the real axis from -2 to $\frac{1}{4}$. In fact, one of the key unresolved problems about the Mandelbrot set M is to decide whether M is not only connected, but also *pathwise* connected, that is, to show that each pair of points in M can be joined by a path lying entirely in M. This is closely linked to another question, namely whether the Mandelbrot set has the property of being *locally connected* (the so-called MLC conjecture), which is in turn linked to the question of whether there exist any interior points of M that are *not* part of a periodic region.



Adrien Douady



Tan Lei

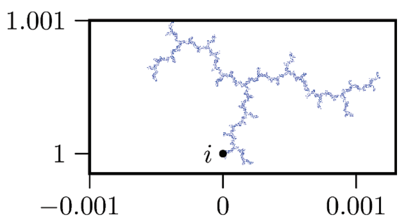


Figure 4.11 The set M near $c = i$

Properties of the Mandelbrot set

Adrien Douady (1935–2006) was a professor at the University of Paris-Sud, where he was known for his great mathematical skill and insight, and his humour – the Julia set in Figure 2.18(b) is known as the **Douady rabbit** after his imaginative description of it. Douady provided a focus for much of the hugely exciting work that took place in the years after the Mandelbrot set was discovered. For example, he and John Hubbard (1945–) not only proved that M is connected but also developed a theory that explains why the set M contains 'baby' Mandelbrot sets.

Another remarkable property of the Mandelbrot set was discovered around 1990 by the Chinese mathematician Tan Lei (1963–2016), who had been a student of Douady and who went on to be a professor at the University of Angers, France. She showed that for certain special points c in the Mandelbrot set, the shape of M near c resembles more and more closely the shape of the Julia set J_c as you zoom in ever closer to c. One such point is $c = i$; see Figure 4.11 for a picture of M near i, and compare this with Figure 2.4(b), showing J_i, which in this case is the same as K_i.

Finally, we describe an effective *graphical* method for obtaining information about the periodic regions of the Mandelbrot set whose centres lie on the real axis. The idea is to choose a large number of real values of c between -2 and $\frac{1}{4}$, and plot each of the corresponding real sequences $x_n = P_c^n(0)$ vertically above the corresponding value of c; see Figure 4.12(a), where the horizontal axis is the c-axis and the vertical axis can be thought of as the x_n-axis.

In order to determine any values of c that have attracting p-cycles (to which such a sequence (x_n) will be attracted, by the remark following Theorem 4.5), the first 200 or so terms are discarded and the next 600 or so are plotted. Thus if P_c has an attracting p-cycle for some real value c, then p points should be plotted above the corresponding point c.

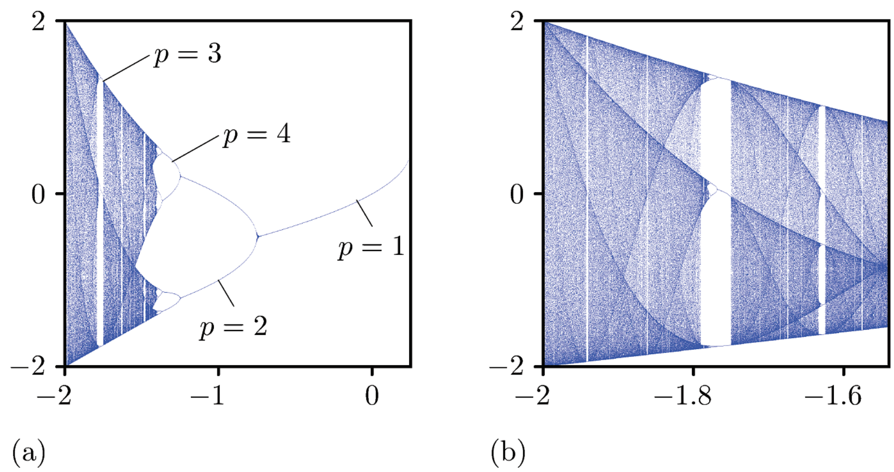


Figure 4.12 Bifurcation diagram for (a) $-2 \le c \le 0.25$, (b) $-2 \le c \le -1.54$

The bifurcation diagram in Figure 4.12(a) reveals the convergence of the sequence $(P_c^n(0))$ to an attracting fixed point for $-\frac{3}{4} < c < \frac{1}{4}$, to an attracting 2-cycle for $-\frac{5}{4} < c < -\frac{3}{4}$, to an attracting 4-cycle for c just to the left of $-\frac{5}{4}$, with further period-doubling bifurcations to the left of this. Also visible is an attracting 3-cycle for c just to the left of $-\frac{7}{4}$.

For other values of c it is less clear what is happening, but by scaling the c-axis appropriately (see Figure 4.12(b)) many other 'periodic windows' are revealed which correspond to attracting p-cycles for other values of p.

In fact, pictures of this type for the related family of iteration sequences given by

$$x_{n+1} = k\,x_n(1 - x_n), \quad \text{with } x_0 = \tfrac{1}{2}, \quad \text{where } 0 \le k \le 4,$$

were studied in the early 1970s, before the Mandelbrot set itself had been plotted. In particular, the order in which the periodic windows appear was found and the rate at which period-doubling occurs was discovered (by the American mathematical physicist Mitchell Feigenbaum) to have a certain universal property. Thus even the part of the Mandelbrot set that lies on the real axis is extremely complicated. For example, it was proved (around 1997) that every non-empty open interval of $\left[-2, \frac{1}{4}\right]$ meets at least one of the periodic windows. Two proofs were given, one by the Ukranian-born mathematician Mikhail Lyubich, and another by the Polish-born mathematicians Jacek Graczyk and Grzegorz Świątek; both proofs are more than 100 pages long!

4.4 Proofs of Theorems 4.2 and 4.5

This subsection is intended for reading only (it will not be assessed).

The aim of this subsection is to give *outline* proofs of Theorems 4.2 and 4.5.

First we will need the concept of the **preimage set** $P_c^{-1}(E)$ of a set E under the function P_c. This is the set consisting of all points that are mapped to E by P_c:

$$P_c^{-1}(E) = \{z : P_c(z) \in E\}.$$

The use of this notation does *not* imply that P_c necessarily has an inverse function.

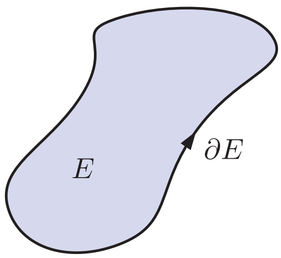


Figure 4.13 A compact disc

For example, if $E = \{-1\}$ and $c = 0$, then $P_0^{-1}(\{-1\}) = \{i, -i\}$. Note that $P_c^{-1}(E)$ is always symmetric under rotation by π about 0, since P_c is an even function.

Next we say that a compact set whose boundary is a simple-closed smooth path is a **compact disc**; see Figure 4.13. Note that a compact disc need not be a disc, but it must be connected. The nature of the preimage set of a compact disc E under P_c is given by the following result (see Figure 4.14). Recall that the interior and exterior of a subset of $\mathbb{C}$ were defined in Subsection 5.1 of Unit A3.

Lemma 4.2

Let E be a compact disc and suppose that $c \notin \partial E$. Then $P_c^{-1}(E)$ is

(a) one compact disc containing 0, if $c \in \operatorname{int} E$

(b) the union of two compact discs, neither containing 0, if $c \in \operatorname{ext} E$.

Note that if $c \in \partial E$, then $P_c^{-1}(E)$ forms a filled-in figure of eight.

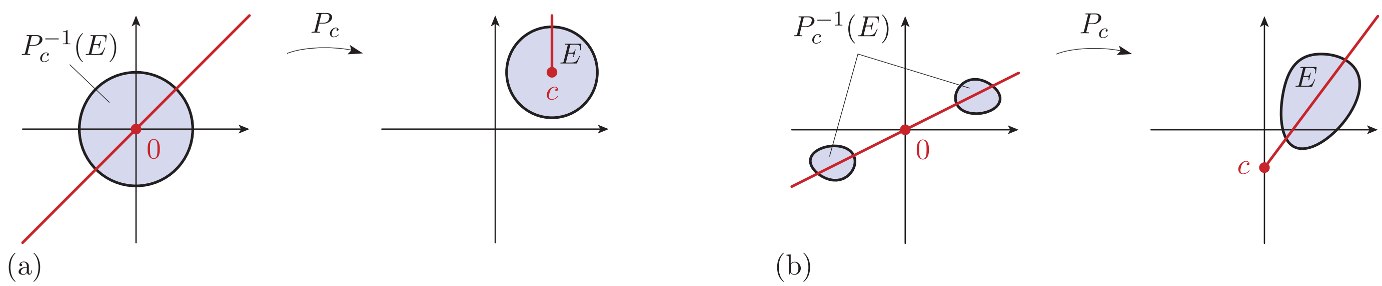


Figure 4.14 Preimage sets of a compact disc when (a) $c \in \operatorname{int} E$, (b) $c \in \operatorname{ext} E$

To indicate why Lemma 4.2 is true, we have shown in both parts of Figure 4.14 a typical ray emerging from the point c and meeting the set E, as well as the preimage of this ray, which consists of the two rays emerging from 0, combining to form a line through 0.

We now outline a proof of Theorem 4.2.

Theorem 4.2 Fatou–Julia Theorem

For any $c \in \mathbb{C}$,

$$K_c \text{ is connected} \iff 0 \in K_c.$$

Proof We start by choosing a closed disc $E_0 = \{z : |z| \le r\}$, such that

$$r > \max\{r_c, |c|\}, \quad \text{where } r_c = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + |c|}, \tag{4.9}$$

and

$$P_c^n(0) \notin \partial E_0, \quad \text{for } n = 1, 2, \ldots. \tag{4.10}$$

Then we define the sequence of successive preimage sets of E_0 under P_c:

$$E_1 = P_c^{-1}(E_0),\ E_2 = P_c^{-1}(E_1),\ \ldots;$$

that is, we define

$$E_n = \{z : P_c^n(z) \in E_0\}, \quad \text{for } n = 1, 2, \ldots.$$

If $P_c^{n+1}(z) \in E_0$, then we must have $P_c^n(z) \in E_0$ also (by Theorem 2.2, because $r > r_c$). Hence

$$E_{n+1} \subseteq E_n, \quad \text{for } n = 0, 1, 2, \ldots,$$

so the sets E_n are nested. Moreover,

$$\begin{aligned} E_0 \cap E_1 \cap E_2 \cap \cdots &= \{z : P_c^n(z) \in E_0,\ \text{for } n = 0, 1, 2, \ldots\} \\ &= \{z : |P_c^n(z)| \le r,\ \text{for } n = 0, 1, 2, \ldots\} \\ &= K_c, \end{aligned} \tag{4.11}$$

by Theorem 2.2, since $r > r_c$. Thus the shape of K_c is determined by the shapes of the sets E_n.

Figure 4.15 illustrates the first few of these nested sets $E_0, E_1, E_2, \ldots$ for the two cases $c = -1$ and $c = 1$. In both cases we have taken $E_0 = \{z : |z| \le r\}$, where $r = 1.8$, so properties (4.9) and (4.10) are satisfied in each case. Points of $E_n - E_{n+1}$ are plotted in blue if n is even and white if n is odd.

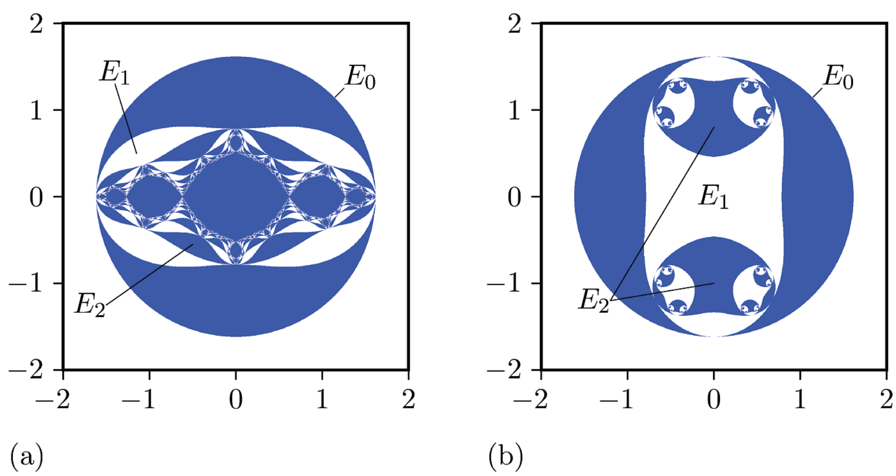


Figure 4.15 The nested sets $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ for (a) $c = -1$, (b) $c = 1$

In Figure 4.15(a) *all* the sets E_n are compact discs. However, in Figure 4.15(b) the sets E_0 and E_1 are compact discs, but E_2 consists of two compact discs, E_3 consists of four compact discs, and so on. We now show that these different structures are related to whether or not the point 0 lies in K_c.

First assume that $0 \in K_c$, so $c = P_c(0) \in K_c$. Then

$$c \in \operatorname{int} E_n, \quad \text{for } n = 0, 1, 2, \ldots,$$

by property (4.10) and equation (4.11). Thus, by Lemma 4.2(a),

$$E_{n+1} = P_c^{-1}(E_n) \text{ is one compact disc}, \quad \text{for } n = 0, 1, 2, \ldots.$$

Thus $E_0, E_1, E_2, \ldots$ is a nested sequence of connected compact sets, as in Figure 4.15(a), and it follows from this (by a result in metric space theory) that $K_c = E_0 \cap E_1 \cap \cdots$ is also connected, as required.

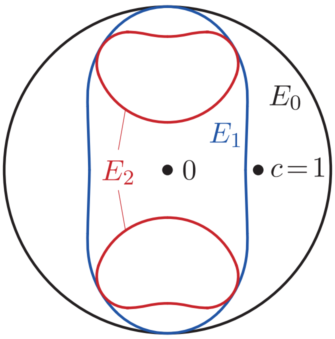


Figure 4.16 Preimage sets of E_0 when $c = 1$

Next assume that $0 \notin K_c$, so $c = P_c(0) \notin K_c$. Since $c \in \operatorname{int} E_0$, there is a positive integer m such that

$$c \in \operatorname{int} E_{m-1}, \quad \text{but } c \notin E_m.$$

Thus, by Lemma 4.2(b), E_m is one compact disc, but E_{m+1} is two; see Figure 4.16, which is based on Figure 4.15(b), in which $c = 1$ and $m = 1$. Thus the set K_c lies in the union of the two halves of E_{m+1}, and it has points in each half because both K_c and E_{m+1} are symmetric under rotation by π about 0. Hence K_c is disconnected, as required. This completes the (outline) proof of Theorem 4.2. ■

We end with a proof of Theorem 4.5, which uses the machinery developed in the previous proof.

Theorem 4.5

If the function P_c has an attracting cycle, then $c \in M$.

Proof The idea of this proof is to show that if $c \notin M$, then K_c has no interior points, so P_c cannot have an attracting cycle, by Theorem 2.5(a). This is a proof by contraposition.

If $c \notin M$, then $0 \notin K_c$ by Theorem 4.2. Hence, by the second part of the proof of Theorem 4.2, there exists m such that

$$c \in \operatorname{int} E_{m-1}, \quad \text{but } c \notin E_m.$$

It follows that c lies outside both halves of E_{m+1}. Hence, by Lemma 4.2(b), the set $E_{m+2} = P_c^{-1}(E_{m+1})$ consists of four compact discs (two for each half of E_{m+1}). In general, E_{m+n} consists of 2^n compact discs. Now, K_c is a subset of each of these preimage sets E_{m+n}, so any connected subset $\widetilde{K}_c$ of K_c must lie in exactly one of the 2^n compact discs comprising E_{m+n}, say $\widetilde{E}_{m+n}$. Thus

$$\widetilde{E}_{m+1} \supseteq \widetilde{E}_{m+2} \supseteq \cdots \supseteq \widetilde{K}_c.$$

It can be proved, by a rather tricky application of the Riemann Mapping Theorem and Schwarz's Lemma, that the diameters of these compact discs $\widetilde{E}_{m+n}$ tend to zero as $n \to \infty$, so $\widetilde{K}_c$ must be a singleton set, and there will be one such singleton set for each of the (infinitely many) possible nested sequences of compact discs $\widetilde{E}_{m+n}$. It follows that if $c \notin M$, then K_c has no interior points, as required. ■

Cantor sets

A set obtained by a construction of the type in the proof of Theorem 4.5 is called a **Cantor set**, named after the German mathematician Georg Cantor (1845–1918) who developed the foundations of the subject of *set theory*.

Cantor proved that the set of rational numbers is countable (that is, its elements can be arranged to form a sequence), but the set of real numbers is uncountable (that is, it is not countable). Thus in a sense there are far more real numbers than rational numbers! It can be shown that any Cantor set is uncountable, so in particular the keep set K_c is uncountable for each c not in M.

Further exercises

Exercise 4.9

Use Theorem 4.3 to determine which of the following points c lie in M.

(a) $c = -1 + 2i$ (b) $c = -1$ (c) $c = -1 + i$

Exercise 4.10

Show that if $|c| = 2$ but $c \neq -2$, then

$$|c^2 + c| > 2.$$

Deduce that $c \notin M$.

(*Hint*: Use the factorisation $c^2 + c = c(c + 1)$.)

Exercise 4.11

Prove that each of the following points lies in M.

(a) $c = -1.1 - 0.1i$ (b) $c = 0.6i$

Exercise 4.12

(a) Prove that if $\mathcal{R}$ is a periodic region of M, then $\partial\mathcal{R} \subseteq M$.

(b) Deduce from part (a) that each of the following points lies in M.

(i) $c = \frac{1}{4} + \frac{1}{2}i$ (ii) $c = -1 + \frac{1}{4}i$

(*Hint*: Locate these points in Figure 4.9.)

Exercise 4.13

Explain why there can be at most six values of c (see Figure 4.9) for which P_c has a super-attracting 4-cycle.

Exercise 4.14

Show that a period-doubling bifurcation occurs at $c = -\frac{5}{4}$, and relate this fact to Figure 4.9.

5 Beyond the Mandelbrot set

This section is intended for reading only (it will not be assessed).

After reading through this section, you should be able to:

- appreciate the universal nature of the Mandelbrot set.

In this section we demonstrate that the Mandelbrot set appears when we iterate various families of functions other than quadratics.

In Subsection 1.4 we briefly discussed the Newton–Raphson function

$$N(z) = \frac{2z^3 + 1}{3z^2}, \tag{5.1}$$

corresponding to the cubic polynomial function $p(z) = z^3 - 1$.

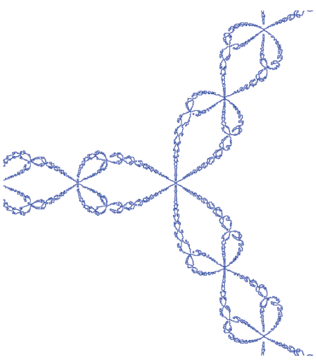


Figure 5.1 The watershed for Newton–Raphson iteration

Under iteration of the function N, all points of $\mathbb{C}$ are attracted to one of the zeros of p, or else they remain on the common basin boundary (see Figure 5.1), which includes the point at ∞.

At about the same time as the discovery of the Mandelbrot set, around 1979, the Newton–Raphson method for a general cubic function was investigated by computer experiments. For most cubic functions, the corresponding Newton–Raphson function behaves under iteration in the same way as the function in equation (5.1), but for some cubic functions a difference was found.

To make this difference precise, we consider the family of cubic functions given by

$$\begin{aligned} p_c(z) &= (z-1)\big(z+\tfrac{1}{2}-c\big)\big(z+\tfrac{1}{2}+c\big) \\ &= z^3 - \big(\tfrac{3}{4}+c^2\big)z - \tfrac{1}{4} + c^2, \end{aligned}$$

where $c \in \mathbb{C}$. For example, if $c = \pm(\sqrt{3}/2)i$, then $p_c(z) = z^3 - 1$. The Newton–Raphson function corresponding to P_c is

$$N_c(z) = z - \frac{p_c(z)}{p_c'(z)} = \frac{2z^3 + \big(\frac{1}{4} - c^2\big)}{3z^2 - \big(\frac{3}{4} + c^2\big)},$$

and it is straightforward to check that the critical points of N_c (that is, the points where N_c' vanishes) are the three zeros of p_c *and* the point 0.

For most values of c, the critical point 0 is attracted to one of the zeros of p_c under iteration of N_c, although it may also remain on the basin boundary; for example, if $c = \pm(\sqrt{3}/2)i$, then $N_c(0) = \infty$ and $N_c(\infty) = \infty$. For some values of c, however, the function N_c has an attracting p-cycle, where $p > 1$, to which the point 0 is attracted. In Figure 5.2(a) we have plotted those values of c in the box $\{c : -2 \le \operatorname{Re} c \le 2,\ -2 \le \operatorname{Im} c \le 2\}$ for which the sequence $(N_c^n(0))$ does *not* converge to one of the zeros of p_c.

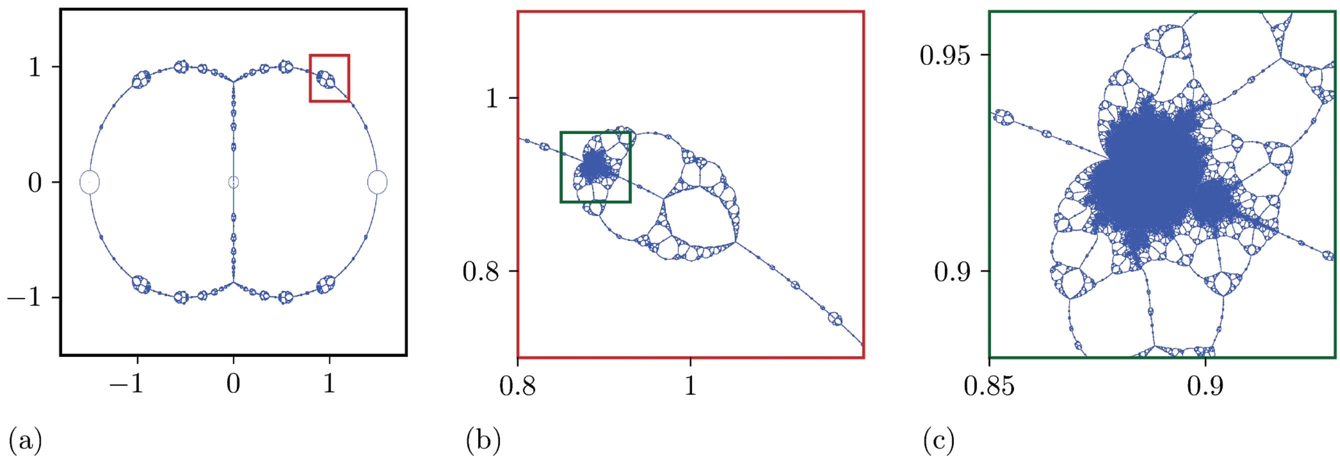


Figure 5.2 (a) The set $\{c : N_c^n(0) \not\to \text{a zero of } p_c\}$, and close-ups in (b) and (c)

If we zoom in on various parts of this set, as shown in (b) and (c) of Figure 5.2, then we find small copies of the Mandelbrot set! It was this type of example that inspired Douady and Hubbard to develop a theory which shows that copies of the Mandelbrot set appear in the parameter plane whenever we consider the iteration of suitable families of analytic functions. In this sense, the Mandelbrot set has a universal nature!

To emphasise this universal nature, we describe one relative of the Mandelbrot set, which is obtained by iterating *non-analytic* functions of the form

$$f_c(z) = \overline{z}^2 + c,$$

where $c \in \mathbb{C}$. By analogy with Theorem 4.3, we plot the set

$$\{c : |f_c^n(0)| \leq 2, \text{ for } n = 0, 1, 2, \dots\},$$

which is called the **tricorn**, or **Mandelbar set**; see Figure 5.3(a).

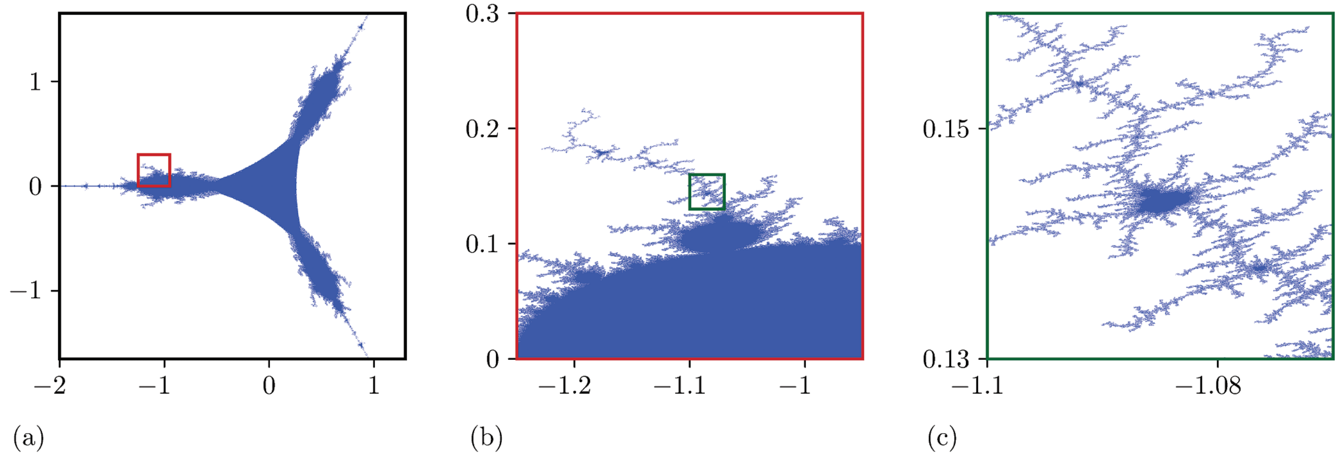


Figure 5.3 (a) The tricorn, and close-ups in (b) and (c)

This set is symmetric under rotation by $2\pi/3$ about 0 (as well as under reflection in the real axis), and it appears to be connected; this property was proved in 1993 by the Japanese mathematician Shizuo Nakane. Closer inspection reveals that some parts of the boundary of the tricorn are smooth (for example, near $c = \frac{1}{4}$), whereas other parts are extremely irregular (see Figure 5.3(b)). As you might expect, the tricorn appears to contain small copies of itself, and it also appears to contain small copies of the Mandelbrot set!

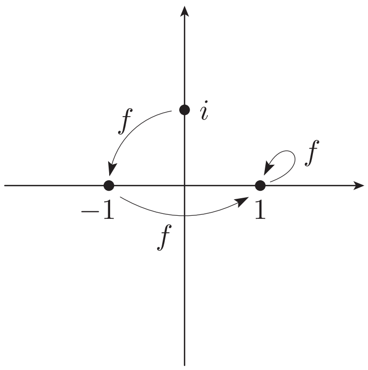
To conclude this unit, we observe that our investigations have only scratched the surface of the subject of complex iteration. For example, there is much more to be said about the structure of the individual Julia sets J_c, and there are many families of entire functions (such as $z \longmapsto e^{cz}$, where c is a complex parameter) whose behaviour under iteration leads to completely new phenomena. Nevertheless, we hope that you have gained some insight into this remarkable subject, and that you appreciate the irony in the following quotation from Adrien Douady:

> I must say that, in 1980, whenever I told my friends that I was just starting with J. H. Hubbard a study of polynomials of degree 2 in one complex variable (and more specifically those of the form $z \longmapsto z^2 + c$), they would all stare at me and ask: Do you expect to find anything new?
>
> (Douady, 1986, p. 161)

Solutions to exercises

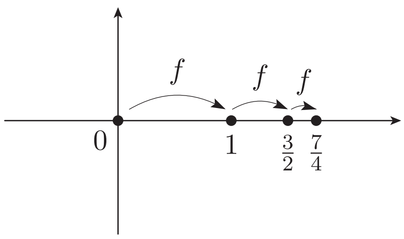
Solution to Exercise 1.1

(a) $z_0 = i,\ z_1 = -1,\ z_2 = 1,\ z_3 = 1.$



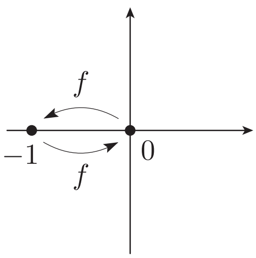
Here $f(z) = z^2$.

(b) $z_0 = 0,\ z_1 = 1,\ z_2 = \frac{3}{2},\ z_3 = \frac{7}{4}.$



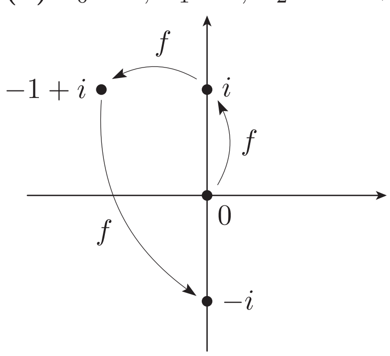
Here $f(z) = \frac{1}{2}z + 1$.

(c) $z_0 = 0,\ z_1 = -1,\ z_2 = 0,\ z_3 = -1.$



Here $f(z) = z^2 - 1$.

(d) $z_0 = 0,\ z_1 = i,\ z_2 = -1 + i,\ z_3 = -i.$



Here $f(z) = z^2 + i$.

Solution to Exercise 1.2

Since $f(z) = \frac{1}{2}z + 1$, we have

$$\begin{aligned} f^2(z) &= f(f(z)) \\ &= f\left(\tfrac{1}{2}z + 1\right) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}z + 1\right) + 1 \\ &= \tfrac{1}{4}z + \tfrac{3}{2}. \end{aligned}$$

Also,

$$\begin{aligned} f^3(z) &= f^2(f(z)) \\ &= \tfrac{1}{4}\left(\tfrac{1}{2}z + 1\right) + \tfrac{3}{2} \\ &= \tfrac{1}{8}z + \tfrac{7}{4}. \end{aligned}$$

Solution to Exercise 1.3

(a) Since $f(z) = z + b$, we have

$$\begin{aligned} f^1(z) &= z + b, \\ f^2(z) &= (z + b) + b = z + 2b, \\ f^3(z) &= (z + 2b) + b = z + 3b, \end{aligned}$$

and, in general,

$$f^n(z) = z + nb, \quad \text{for } n = 1, 2, \ldots .$$

(b) Since $f(z) = z^3$, we have

$$\begin{aligned} f^1(z) &= z^3, \\ f^2(z) &= (z^3)^3 = z^9, \\ f^3(z) &= (z^9)^3 = z^{27}, \end{aligned}$$

and, in general,

$$f^n(z) = z^{3^n}, \quad \text{for } n = 1, 2, \ldots .$$

Solution to Exercise 1.4

(a) If $z_0 = 1$, then we have

$$z_n = 1, \quad \text{for } n = 1, 2, \ldots,$$

so (z_n) is a constant sequence, and converges to 1.

(b) If $z_0 = -i$, then the terms of the sequence are

$$-i, -1, 1, 1, 1, \ldots,$$

so (z_n) is eventually constant, and converges to 1.

(c) If $z_0 = e^{2\pi i/3}$, then the terms of the sequence are

$$e^{2\pi i/3}, e^{4\pi i/3}, e^{8\pi i/3} = e^{2\pi i/3}, \dots,$$

so the sequence cycles endlessly between these two values.

(d) Since $z_n = z_0^{2^n}$ and $|1/z_0| < 1$, we deduce that

$$\frac{1}{z_n} = \left(\frac{1}{z_0}\right)^{2^n}, \quad n = 1, 2, \dots,$$

is a null sequence. Hence $z_n \to \infty$ as $n \to \infty$, by the Reciprocal Rule (Theorem 1.5 of Unit A3).

Solution to Exercise 1.5

(a) The fixed point equation is $f(z) = \frac{1}{2}z + 1 = z$, and

$$\tfrac{1}{2}z + 1 = z \iff \tfrac{1}{2}z = 1,$$

so the only fixed point is 2.

(b) The fixed point equation is $f(z) = z^2 - 2 = z$, and

$$\begin{aligned} z^2 - 2 = z &\iff z^2 - z - 2 = 0 \\ &\iff (z-2)(z+1) = 0, \end{aligned}$$

so the only fixed points are 2 and -1.

(c) The fixed point equation is $f(z) = z^3 = z$, and

$$z^3 = z \iff z(z^2 - 1) = 0,$$

so the only fixed points are 0, 1 and -1.

Solution to Exercise 1.6

(a) Since $f'(z) = 2z$, we have

$$|f'(0)| = 0 < 1,$$

so 0 is an attracting (in fact, super-attracting) fixed point of f, and

$$|f'(1)| = 2 > 1,$$

so 1 is a repelling fixed point of f.

(b) Since $f'(z) = \frac{1}{2}$, we have

$$|f'(2)| = \tfrac{1}{2} < 1,$$

so 2 is an attracting fixed point of f.

(c) Since $f'(z) = 2z$, we have

$$|f'(2)| = 4 > 1,$$

so 2 is a repelling fixed point of f.

Solution to Exercise 1.7

(a) From Example 1.2(a) we know that

$$f^n(z) = \left(\tfrac{1}{2}\right)^n z, \quad \text{for } n = 1, 2, \dots.$$

Hence

$$f^n(z) \to 0 \text{ as } n \to \infty, \quad \text{for all } z \in \mathbb{C}.$$

Therefore the basin of attraction of 0 under f is

$$\{z : f^n(z) \to 0 \text{ as } n \to \infty\} = \mathbb{C}.$$

(b) From Exercise 1.3(b) we know that

$$f^n(z) = z^{3^n}, \quad \text{for } n = 1, 2, \dots.$$

Hence

$$f^n(z_0) \to 0 \text{ as } n \to \infty, \quad \text{for } |z_0| < 1,$$

but

$$f^n(z_0) \not\to 0 \text{ as } n \to \infty, \quad \text{for } |z_0| \geq 1,$$

since $|f^n(z_0)| = |z_0|^{3^n} \geq 1$, for $n \geq 1$.

Thus the basin of attraction of 0 under f is

$$\{z : f^n(z) \to 0 \text{ as } n \to \infty\} = \{z : |z| < 1\}.$$

Solution to Exercise 1.8

First we note that the function $h(z) = -z + \frac{1}{2}$ is one-to-one on $\mathbb{C}$. Putting

$$w_n = h(z_n) = -z_n + \tfrac{1}{2},$$

we have $z_n = -w_n + \frac{1}{2}$, so

$$z_{n+1} = z_n - z_n^2, \quad n = 0, 1, 2, \dots,$$

becomes

$$\begin{aligned} -w_{n+1} + \tfrac{1}{2} &= \left(-w_n + \tfrac{1}{2}\right) - \left(-w_n + \tfrac{1}{2}\right)^2 \\ &= -w_n + \tfrac{1}{2} - w_n^2 + w_n - \tfrac{1}{4}; \end{aligned}$$

that is,

$$w_{n+1} = w_n^2 + \tfrac{1}{4}, \quad n = 0, 1, 2, \dots.$$

This proves that the sequences (z_n) and (w_n) are conjugate iteration sequences with conjugating function $h(z) = -z + \frac{1}{2}$.

If $z_0 = \frac{1}{2}$, then $w_0 = 0$.

Solution to Exercise 1.9

(a) First we note that the function

$$h(z) = z + b/(a-1),$$

where $a \neq 1$, is one-to-one on $\mathbb{C}$.

Putting $w_n = h(z_n) = z_n + b/(a-1)$, we have $z_n = w_n - b/(a-1)$, so

$$z_{n+1} = az_n + b, \quad n = 0, 1, 2, \ldots,$$

becomes

$$w_{n+1} - b/(a-1) = a(w_n - b/(a-1)) + b;$$

that is,

$$w_{n+1} = aw_n, \quad n = 0, 1, 2, \ldots, \tag{S1}$$

since $-b/(a-1) = -ab/(a-1) + b$. This proves that the sequences (z_n) and (w_n) are conjugate iteration sequences with conjugating function $h(z) = z + b/(a-1)$.

(b) The iteration sequence (S1) has general term $w_n = a^n w_0$ (see Example 1.2(a)). Hence, for $n = 0, 1, 2, \ldots,$

$$z_n = w_n - \frac{b}{a-1} = a^n w_0 - \frac{b}{a-1},$$

giving

$$z_n = a^n\left(z_0 + \frac{b}{a-1}\right) - \frac{b}{a-1};$$

that is,

$$z_n = a^n z_0 + \frac{b(a^n - 1)}{a-1}, \quad \text{for } n = 0, 1, 2, \ldots.$$

(i) If $|a| < 1$, then (a^n) is a null sequence, so

$$z_n \to -\frac{b}{a-1} \text{ as } n \to \infty.$$

(Note that $-b/(a-1)$ is the only fixed point of the function $f(z) = az + b$.)

(ii) If $|a| = 1$, $a \neq 1$, then (a^n) is divergent, by Theorem 1.7(b) of Unit A3. It follows that (z_n) is divergent in this case (unless $z_0 = -b/(a-1)$, in which case the sequence is constant).

(iii) If $|a| > 1$, then (a^n) tends to infinity, by Theorem 1.7(a) of Unit A3. It follows that (z_n) tends to infinity in this case (unless $z_0 = -b/(a-1)$, in which case the sequence is constant).

Solution to Exercise 1.10

With $N(z) = \dfrac{z^2 - b}{2z + a}$ and $h(z) = \dfrac{z - \alpha}{z - \beta}$, we have

$$\begin{aligned}
h(N(z)) &= \left(\frac{z^2 - b}{2z + a} - \alpha\right) \Big/ \left(\frac{z^2 - b}{2z + a} - \beta\right) \\
&= \frac{z^2 - b - \alpha(2z + a)}{z^2 - b - \beta(2z + a)} \\
&= \frac{z^2 - 2\alpha z - (a\alpha + b)}{z^2 - 2\beta z - (a\beta + b)} \\
&= \frac{z^2 - 2\alpha z + \alpha^2}{z^2 - 2\beta z + \beta^2} \quad \text{(by the hint)} \\
&= \frac{(z - \alpha)^2}{(z - \beta)^2} \\
&= (h(z))^2,
\end{aligned}$$

as required.

Solution to Exercise 1.11

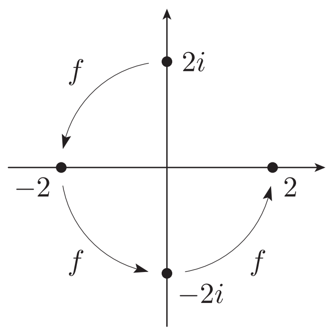
If $p(z) = (z - \alpha)^2$, then

$$N(z) = z - \frac{(z - \alpha)^2}{2(z - \alpha)} = \tfrac{1}{2}(z + \alpha),$$

so, by Exercise 1.9(b)(i) (with $a = \frac{1}{2}, b = \frac{1}{2}\alpha$), all points of $\mathbb{C}$ are attracted to α under iteration of N.

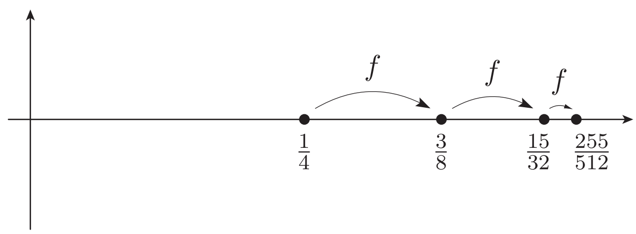
Solution to Exercise 1.12

(a) $z_0 = 2i,\ z_1 = -2,\ z_2 = -2i,\ z_3 = 2.$



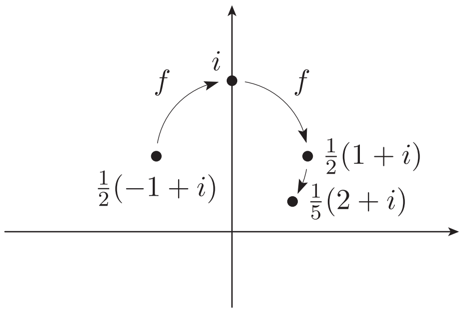
Here $f(z) = iz$.

(b) $z_0 = \frac{1}{4},\ z_1 = \frac{3}{8},\ z_2 = \frac{15}{32},\ z_3 = \frac{255}{512}.$



Here $f(z) = 2z(1 - z)$.

(c) $z_0 = \frac{1}{2}(-1+i)$, $z_1 = i$, $z_2 = \frac{1}{2}(1+i)$, $z_3 = \frac{1}{5}(2+i)$.



Here $f(z) = \dfrac{z}{z+1}$.

Solution to Exercise 1.13

(a) We have to solve $f(z) = z$:

$$\begin{aligned} z - z^2 = z &\iff z^2 = 0 \\ &\iff z = 0, \end{aligned}$$

so the only fixed point of f is 0. Since $f'(z) = 1 - 2z$, we have

$$f'(0) = 1.$$

Thus the fixed point 0 is indifferent.

(b) We have to solve $f(z) = z$:

$$2z(1-z) = z \iff z(1-2z) = 0,$$

so the fixed points of f are 0 and $\frac{1}{2}$. Since $f'(z) = 2 - 4z$, we have

$$f'(0) = 2 \quad \text{and} \quad f'\left(\tfrac{1}{2}\right) = 0.$$

Thus the fixed point 0 is repelling, and the fixed point $\frac{1}{2}$ is super-attracting.

(c) We have to solve $f(z) = z$:

$$z^2 - \tfrac{1}{2} = z \iff z^2 - z - \tfrac{1}{2} = 0,$$

so the fixed points of f are $\frac{1}{2}(1 \pm \sqrt{3})$. Since $f'(z) = 2z$, we have

$$\left|f'\left(\tfrac{1}{2}(1+\sqrt{3})\right)\right| = 1 + \sqrt{3} = 2.732\ldots$$

and

$$\left|f'\left(\tfrac{1}{2}(1-\sqrt{3})\right)\right| = |1 - \sqrt{3}| = 0.732\ldots.$$

Thus the fixed point $\frac{1}{2}(1+\sqrt{3})$ of f is repelling, and the fixed point $\frac{1}{2}(1-\sqrt{3})$ is attracting.

(d) We have to solve $f(z) = z$:

$$\frac{z}{z+1} = z \iff z = 0,$$

so the only fixed point of f is 0. Since

$$f'(z) = \frac{(z+1)\times 1 - z \times 1}{(z+1)^2} = \frac{1}{(z+1)^2},$$

we have

$$f'(0) = 1.$$

Thus the fixed point 0 is indifferent.

Solution to Exercise 1.14

Putting $w_n = h(z_n) = 1 - 2z_n$, for $n = 0, 1, 2, \ldots$, we obtain

$$z_n = \tfrac{1}{2}(1 - w_n), \quad \text{for } n = 0, 1, 2, \ldots.$$

Hence the iteration sequence (z_n) is conjugate to the iteration sequence (w_n), where

$$\begin{aligned} \tfrac{1}{2}(1 - w_{n+1}) &= 2\left(\tfrac{1}{2}(1-w_n)\right)\left(1 - \tfrac{1}{2}(1-w_n)\right) \\ &= \tfrac{1}{2}(1-w_n)(1+w_n) \\ &= \tfrac{1}{2}(1-w_n^2), \end{aligned}$$

so $w_{n+1} = w_n^2$, for $n = 0, 1, 2, \ldots$, as required.

Since $w_n = w_0^{2^n}$, for $n = 0, 1, 2, \ldots$, by Example 1.2(b), we deduce that

$$\begin{aligned} z_n &= \tfrac{1}{2}(1 - w_n) \\ &= \tfrac{1}{2}\left(1 - w_0^{2^n}\right) \\ &= \tfrac{1}{2}\left(1 - (1-2z_0)^{2^n}\right), \quad \text{for } n = 0, 1, 2, \ldots. \end{aligned}$$

Solution to Exercise 2.1

(a) By Theorem 2.1, with $a = -4$, $b = 4$, $c = 0$, the iteration sequence

$$z_{n+1} = -4z_n^2 + 4z_n, \quad n = 0, 1, 2, \ldots,$$

is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2 = 2 - 4 = -2$, with conjugating function

$$h(z) = az + \tfrac{1}{2}b = -4z + 2.$$

In this case $w_0 = h\left(\frac{1}{2}\right) = 0$.

(b) By Theorem 2.1, with $a = -2$, $b = 0$, $c = 1$, the iteration sequence

$$z_{n+1} = 1 - 2z_n^2, \quad n = 0, 1, 2, \ldots,$$

is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2 = -2$, with conjugating function

$$h(z) = az + \tfrac{1}{2}b = -2z.$$

In this case $w_0 = h(0) = 0$.

Solution to Exercise 2.2

(a) We have

$$\begin{aligned} P_c^2(z) &= P_c(P_c(z)) \\ &= (z^2 + c)^2 + c \\ &= z^4 + 2cz^2 + c^2 + c, \\ P_c^3(z) &= (z^4 + 2cz^2 + c^2 + c)^2 + c \\ &= z^8 + 4cz^6 + (6c^2 + 2c)z^4 \\ &\quad + (4c^3 + 4c^2)z^2 + c^4 + 2c^3 + c^2 + c. \end{aligned}$$

(b) We have

$$P_c^{n+1}(z) = P_c(P_c^n(z)) = (P_c^n(z))^2 + c. \tag{S2}$$

Since $P_c(z) = z^2 + c$, it follows from equation (S2) that the degree of P_c^{n+1} is twice the degree of P_c^n, and hence the degree of P_c^n is 2^n.

Now P_c is an even function, because

$$P_c(-z) = (-z)^2 + c = z^2 + c = P_c(z),$$

so the fact that P_c^n is an even function follows from equation (S2) by the Principle of Mathematical Induction.

Solution to Exercise 2.3

(a) The fixed point equation is $P_c(z) = z^2 + c = z$, and

$$z^2 + c = z \iff z^2 - z + c = 0.$$

Thus the fixed points are

$$\tfrac{1}{2}(1 \pm \sqrt{1 - 4c}) = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - c}.$$

Suppose that $c \neq \frac{1}{4}$. Then the two complex numbers $\sqrt{1 - 4c}$ and $-\sqrt{1 - 4c}$ are non-zero.

Here $\sqrt{1 - 4c}$ means the principal square root of $1 - 4c$, as usual, so $\sqrt{1 - 4c}$ has a non-negative real part.

Hence $\alpha = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$ lies in $\{z = x + iy : x \geq \frac{1}{2}\}$ and is not equal to $\frac{1}{2}$. Then $P_c'(\alpha) = 2\alpha$ lies in $\{z = x + iy : x \geq 1\}$ and is not equal to 1. This implies that $P_c'(\alpha)$ lies outside the unit circle, so the corresponding fixed point $\alpha = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$ is repelling.

(b) If $c = \frac{1}{4}$, then $P_c = P_{1/4}$ has just one fixed point $\frac{1}{2}$, which is indifferent since $P_{1/4}'(\frac{1}{2}) = 1$.

Solution to Exercise 2.4

(a) We have

$$\begin{aligned} r_0 &= \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + 0} = \tfrac{1}{2} + \tfrac{1}{2} = 1, \\ r_i &= \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + |i|} = \tfrac{1}{2} + \sqrt{\tfrac{5}{4}} = \tfrac{1}{2}(1 + \sqrt{5}), \\ r_{-2} &= \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + |-2|} = \tfrac{1}{2} + \sqrt{\tfrac{9}{4}} = 2. \end{aligned}$$

(b) Since $P_0(z) = z^2$, we have $P_0(1) = 1$. So the sequence $(P_0^n(1))$ has terms $1, 1, 1, \ldots$, and hence $P_0^n(1) \not\to \infty$ as $n \to \infty$.

This sequence shows that, in the case $c = 0$, Theorem 2.2 does not hold if r_0 is replaced by a smaller value.

Since $P_{-2}(z) = z^2 - 2$, we have $P_{-2}(2) = 4 - 2 = 2$. So the sequence $(P_{-2}^n(2))$ has terms $2, 2, 2, \ldots$, and hence $P_{-2}^n(2) \not\to \infty$ as $n \to \infty$.

This sequence shows that, in the case $c = -2$, Theorem 2.2 does not hold if r_{-2} is replaced by a smaller value.

Solution to Exercise 2.5

(a) **(i)** If $z_0 \in L$, then z_0 is real with $|z_0| \leq 2$, so

$$z_1 = z_0^2 - 2$$

is real with $-2 \leq z_1 \leq 2$ because $0 \leq z_0^2 \leq 4$; hence $z_1 \in L$. On repeating this process we deduce that $z_n \in L$, for $n = 1, 2, \ldots$.

(ii) If $z_0 \in \mathbb{C} - L$, then $z_1 \in \mathbb{C} - L$. For if $z_1 \in L$, then

$$z_0^2 = z_1 + 2 \in [0, 4] \implies z_0 \in L.$$

On repeating this process we deduce that

$$z_n \in \mathbb{C} - L, \quad \text{for } n = 1, 2, \ldots.$$

(b) Let $w_n = J^{-1}(z_n)$, for $n = 0, 1, 2, \ldots$, so

$$|w_n| > 1, \quad \text{for } n = 0, 1, 2, \ldots,$$

by the hint. Then $z_n = J(w_n)$ and so, for $n = 0, 1, 2, \ldots$, the equation $z_{n+1} = z_n^2 - 2$ becomes

$$\begin{aligned} J(w_{n+1}) &= (J(w_n))^2 - 2 \\ &= \left(w_n + \frac{1}{w_n}\right)^2 - 2 \\ &= w_n^2 + \frac{1}{w_n^2} = J(w_n^2). \end{aligned}$$

Now, the function J is one-to-one on $\{w : |w| > 1\}$ and hence

$$w_{n+1} = w_n^2, \quad \text{for } n = 0, 1, 2, \ldots.$$

Since $|w_0| > 1$, we have $w_n \to \infty$ as $n \to \infty$, so $z_n = w_n + 1/w_n \to \infty$ as $n \to \infty$ (by the Reciprocal Rule and Combination Rules for sequences).

(c) By part (a) we find that no point of L belongs to E_{-2}, whereas by part (b) all points of $\mathbb{C} - L$ belong to E_{-2}. Hence $E_{-2} = \mathbb{C} - L$ and $K_{-2} = \mathbb{C} - E_{-2} = L$.

Solution to Exercise 2.6

To prove that E_c is completely invariant under P_c, we note that

$$\begin{aligned} z \in E_c &\iff P_c^n(z) \to \infty \text{ as } n \to \infty \\ &\iff P_c^{n+1}(z) \to \infty \text{ as } n \to \infty \\ &\iff P_c^n(P_c(z)) \to \infty \text{ as } n \to \infty \\ &\iff P_c(z) \in E_c, \end{aligned}$$

as required.

Solution to Exercise 2.7

(a) Since

$$P_i(-i) = (-i)^2 + i = -1 + i$$

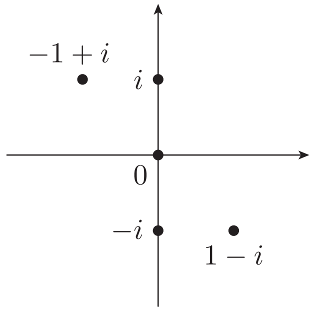
and

$$P_i(-1 + i) = (-1 + i)^2 + i = -i,$$

we have

$$P_i^2(-i) = -i \quad \text{and} \quad P_i^2(-1 + i) = -1 + i,$$

so $-i$ and $-1 + i$ form a 2-cycle of P_i. Hence both these points lie in K_i, as do i and $1 - i$, by Theorem 2.3(e). Another point lying in K_i is 0 because $P_i(0) = i$ and $i \in K_i$, and K_i is completely invariant under P_i.



Clearly none of these points is a fixed point of P_i.

(b) Since $P_0^3(z) = z^8$, we have to solve the equation $z^8 = z$:

$$\begin{aligned} z^8 = z &\iff z^8 - z = 0 \\ &\iff z(z^7 - 1) = 0. \end{aligned}$$

The solutions are 0 and $e^{2\pi k i/7}$, $k = 0, 1, \ldots, 6$ (by Theorem 3.1 of Unit A1).

Of these, the points 0 and 1 (corresponding to $k = 0$) are fixed points of P_0, whereas

$$\begin{aligned} P_0(e^{2\pi i/7}) &= e^{4\pi i/7}, \\ P_0(e^{4\pi i/7}) &= e^{8\pi i/7}, \\ P_0(e^{8\pi i/7}) &= e^{16\pi i/7} = e^{2\pi i/7}, \end{aligned}$$

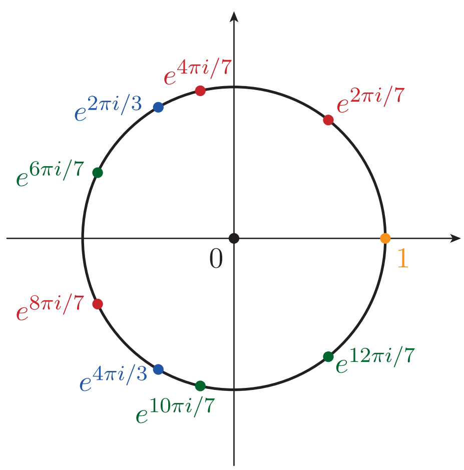
and

$$\begin{aligned} P_0(e^{6\pi i/7}) &= e^{12\pi i/7}, \\ P_0(e^{12\pi i/7}) &= e^{24\pi i/7} = e^{10\pi i/7}, \\ P_0(e^{10\pi i/7}) &= e^{20\pi i/7} = e^{6\pi i/7}. \end{aligned}$$

Hence

$$e^{2\pi i/7}, e^{4\pi i/7}, e^{8\pi i/7} \quad \text{and} \quad e^{6\pi i/7}, e^{12\pi i/7}, e^{10\pi i/7}$$

are both 3-cycles of P_0. Using this and the 2-cycle $e^{2\pi i/3}, e^{4\pi i/3}$ (from Example 2.1), we obtain the following diagram.



(c) Since

$$\begin{aligned}P_{-5/4}\left(\tfrac{1}{2}(-1+\sqrt{2})\right) &= \left(\tfrac{1}{2}(-1+\sqrt{2})\right)^2 - \tfrac{5}{4}\\ &= \tfrac{1}{4}(3-2\sqrt{2}) - \tfrac{5}{4}\\ &= \tfrac{1}{2}(-1-\sqrt{2})\\ &\neq \tfrac{1}{2}(-1+\sqrt{2})\end{aligned}$$

and

$$\begin{aligned}P^2_{-5/4}\left(\tfrac{1}{2}(-1+\sqrt{2})\right) &= P_{-5/4}\left(\tfrac{1}{2}(-1-\sqrt{2})\right)\\ &= \left(\tfrac{1}{2}(-1-\sqrt{2})\right)^2 - \tfrac{5}{4}\\ &= \tfrac{1}{4}(3+2\sqrt{2}) - \tfrac{5}{4}\\ &= \tfrac{1}{2}(-1+\sqrt{2}),\end{aligned}$$

the point $\frac{1}{2}(-1+\sqrt{2})$ is a periodic point, with period 2, of $P_{-5/4}$. (It forms a 2-cycle, along with $\frac{1}{2}(-1-\sqrt{2})$.)

Solution to Exercise 2.8

(a) The numbers $-i$, $-1+i$ form a 2-cycle of P_i with multiplier

$$\begin{aligned}(P_i^2)'(-i) &= P_i'(-i)P_i'(-1+i)\\ &= (-2i)(-2+2i)\\ &= 4+4i,\end{aligned}$$

by Theorem 2.4(a). Hence

$$\left|(P_i^2)'(-i)\right| = 4\sqrt{2} > 1,$$

so this 2-cycle is repelling.

(b) The numbers $e^{2\pi i/7}$, $e^{4\pi i/7}$, $e^{8\pi i/7}$ form a 3-cycle of P_0 with multiplier

$$\begin{aligned}(P_0^3)'(e^{2\pi i/7}) &= P_0'(e^{2\pi i/7})P_0'(e^{4\pi i/7})P_0'(e^{8\pi i/7})\\ &= (2e^{2\pi i/7})(2e^{4\pi i/7})(2e^{8\pi i/7})\\ &= 8,\end{aligned}$$

by Theorem 2.4(a). Hence

$$\left|(P_0^3)'(e^{2\pi i/7})\right| > 1,$$

so this 3-cycle is repelling.

(c) The numbers $\frac{1}{2}(-1+\sqrt{2})$, $\frac{1}{2}(-1-\sqrt{2})$ form a 2-cycle of $P_{-5/4}$ with multiplier

$$\begin{aligned}(P^2_{-5/4})'\left(\tfrac{1}{2}(-1+\sqrt{2})\right) &= P'_{-5/4}\left(\tfrac{1}{2}(-1+\sqrt{2})\right)\\ &\qquad \times P'_{-5/4}\left(\tfrac{1}{2}(-1-\sqrt{2})\right)\\ &= (-1+\sqrt{2})(-1-\sqrt{2})\\ &= -1,\end{aligned}$$

by Theorem 2.4(a). Hence

$$\left|(P^2_{-5/4})'\left(\tfrac{1}{2}(-1+\sqrt{2})\right)\right| = 1,$$

so this 2-cycle is indifferent.

Solution to Exercise 2.9

Using Theorem 2.1, we find that

$$\begin{aligned}z_{n+1} &= 3z_n(1-z_n)\\ &= -3z_n^2 + 3z_n, \quad n = 0,1,2,\ldots,\end{aligned}$$

$z_0 = \frac{1}{2}$, is conjugate to

$$w_{n+1} = w_n^2 + d, \quad n = 0,1,2,\ldots,$$

where $w_n = -3z_n + \frac{3}{2}$, for $n = 0,1,2,\ldots$, with $w_0 = 0$ and

$$d = -3\times 0 + \tfrac{3}{2} - \tfrac{9}{4} = -\tfrac{3}{4}.$$

Solution to Exercise 2.10

(a) If $|c| \le \frac{1}{4}$ and $|z| \le \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$, then

$$\begin{aligned}|P_c(z)| &= |z^2 + c|\\ &\le |z|^2 + |c| \quad \text{(by the Triangle Inequality)}\\ &\le \left(\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}\right)^2 + |c|\\ &= \tfrac{1}{4} + \sqrt{\tfrac{1}{4} - |c|} + \tfrac{1}{4} - |c| + |c|\\ &= \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}.\end{aligned}$$

(b) If $|c| \le \frac{1}{4}$ and $|z| \le \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$, then, by part (a) applied repeatedly,

$$|P_c^n(z)| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}, \quad \text{for } n = 0,1,2,\ldots,$$

so $P_c^n(z) \not\to \infty$ as $n \to \infty$, and hence $z \notin E_c$.

Therefore if $|c| \le \frac{1}{4}$, then

$$\left\{z : |z| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}\right\} \subseteq K_c.$$

(c) If $|c| \le \frac{1}{4}$, then, by part (b) and Theorem 2.3(a),

$$\begin{aligned}\left\{z : |z| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}\right\} &\subseteq K_c\\ &\subseteq \left\{z : |z| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + |c|}\right\}.\end{aligned}$$

Now, if c is close to 0, then $\frac{1}{2} + \sqrt{\frac{1}{4} \pm |c|}$ are both close to 1, so K_c is approximately equal to the closed unit disc.

Solution to Exercise 2.11

(a) Since

$$f(i) = -i \quad \text{and} \quad f(-i) = i,$$

the point $\alpha = i$ is periodic with period 2, and belongs to the 2-cycle $i, -i$. Since $f'(z) = -1$, the multiplier of this 2-cycle is, by Theorem 2.4(a),

$$f'(i)f'(-i) = (-1) \times (-1) = 1.$$

Hence i is an indifferent periodic point of f with period 2.

(b) Since

$$f\left(\tfrac{1}{2}(-1+\sqrt{5})\right) = \tfrac{1}{2}(-1-\sqrt{5})$$

and

$$f\left(\tfrac{1}{2}(-1-\sqrt{5})\right) = \tfrac{1}{2}(-1+\sqrt{5}),$$

the point $\alpha = \frac{1}{2}(-1+\sqrt{5})$ is periodic with period 2, and belongs to the 2-cycle $\frac{1}{2}(-1+\sqrt{5})$, $\frac{1}{2}(-1-\sqrt{5})$. Since $f'(z) = 2z$, the multiplier of this 2-cycle is, by Theorem 2.4(a),

$$\begin{aligned} &f'\left(\tfrac{1}{2}(-1+\sqrt{5})\right)f'\left(\tfrac{1}{2}(-1-\sqrt{5})\right) \\ &= (-1+\sqrt{5})(-1-\sqrt{5}) = -4. \end{aligned}$$

Hence $\frac{1}{2}(-1+\sqrt{5})$ is a repelling periodic point of f with period 2.

(c) Since

$$f(0) = i \quad \text{and} \quad f(i) = 0,$$

the point $\alpha = 0$ is periodic with period 2, and belongs to the 2-cycle $0, i$. Since $f'(z) = 3z^2$, the multiplier of this 2-cycle is, by Theorem 2.4(a),

$$f'(0)f'(i) = 0.$$

Hence 0 is a super-attracting periodic point of f with period 2.

(d) Since

$$f(e^{\pi i/13}) = e^{3\pi i/13}, \quad f(e^{3\pi i/13}) = e^{9\pi i/13}$$

and

$$f(e^{9\pi i/13}) = e^{27\pi i/13} = e^{\pi i/13},$$

the point $\alpha = e^{\pi i/13}$ is periodic with period 3, and belongs to the 3-cycle $e^{\pi i/13}, e^{3\pi i/13}, e^{9\pi i/13}$.

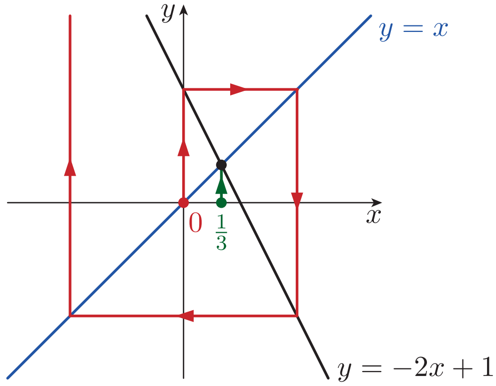
Since $f'(z) = 3z^2$, the multiplier of the 3-cycle is, by Theorem 2.4(a),

$$\begin{aligned} &f'(e^{\pi i/13})f'(e^{3\pi i/13})f'(e^{9\pi i/13}) \\ &= (3e^{2\pi i/13})(3e^{6\pi i/13})(3e^{18\pi i/13}) \\ &= 27e^{26\pi i/13} \\ &= 27. \end{aligned}$$

Hence $e^{\pi i/13}$ is a repelling periodic point of f with period 3.

Solution to Exercise 3.1

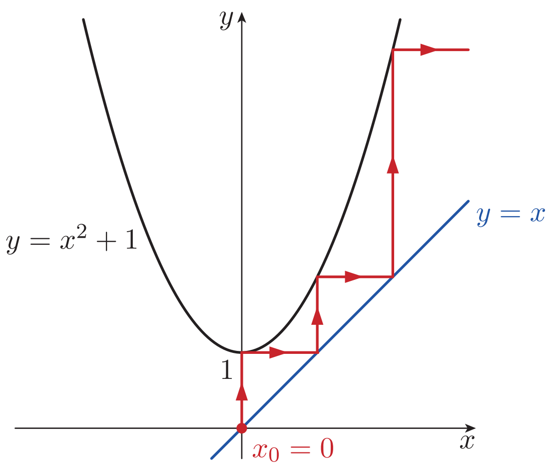
(a) Graphical iteration gives the following diagram.



(b) If $x_0 = 0$, then (x_n) tends to infinity and if $x_0 = \frac{1}{3}$, then (x_n) is constant. In Exercise 1.9(b)(iii) we found that $z_{n+1} = az_n + b$, $n = 0, 1, 2, \ldots$, tends to infinity if $|a| > 1$ unless $z_0 = b/(1-a)$. Here we have $a = -2$, $b = 1$ and $b/(1-a) = \frac{1}{3}$, so our answer agrees with this result.

Solution to Exercise 3.2

(a) With $x_0 = 0$, graphical iteration gives the following diagram.



(b) Observe that

$$x^2 + 1 > x, \quad \text{for all } x \in \mathbb{R},$$

since

$$\begin{aligned} x^2 + 1 > x &\iff x^2 - x + 1 > 0 \\ &\iff \left(x - \tfrac{1}{2}\right)^2 + \tfrac{3}{4} > 0. \end{aligned}$$

Hence the graph of $y = x^2 + 1$ lies strictly above the graph of $y = x$, so the function $f(x) = x^2 + 1$ has no real fixed points. Also, $x_{n+1} = x_n^2 + 1 > x_n$, so the sequence (x_n) is increasing and must tend to infinity because there are no fixed points to prevent this. (An algebraic proof of this fact is given after Lemma 3.1.)

(c) Since $x_n = P_1^n(x_0)$, for $n = 1, 2, \ldots$, we deduce that

$$P_1^n(x_0) \to \infty \text{ as } n \to \infty, \quad \text{for all } x_0 \in \mathbb{R},$$

so no point of $\mathbb{R}$ belongs to K_1.

Solution to Exercise 3.3

(a) If c is real and y is real, then

$$P_c(iy) = (iy)^2 + c = -y^2 + c,$$

which is real.

(b) Suppose that $c > \frac{1}{4}$ and $y \in \mathbb{R}$. By Theorem 3.1, we know that $x \in E_c$ for all $x \in \mathbb{R}$. Hence, by part (a), we have $P_c(iy) \in E_c$. Therefore $iy \in E_c$, by the complete invariance of E_c under P_c, so no point of the imaginary axis belongs to K_c.

Solution to Exercise 3.4

(a) We have $P_c(iy) = -y^2 + c \leq c$.

(b) Suppose that $c < -2$ and $y \in \mathbb{R}$. Then $|c| > 2$, so we see from the given property

$$|c| \leq 2 \iff |c| \leq r_c$$

that $|c| > r_c$, so

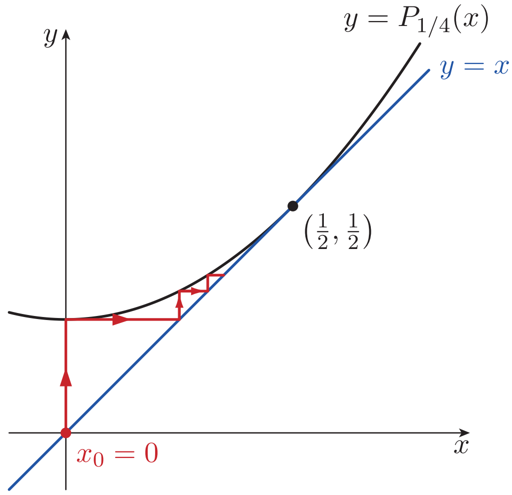
$$c = -|c| < -r_c.$$

Therefore, by part (a),

$$P_c(iy) \leq c < -r_c.$$

Hence $|P_c(iy)| > r_c$, so it follows from Theorem 2.2 that $P_c(iy)$ lies in E_c, and hence $iy \in E_c$ by the complete invariance of E_c under P_c. Therefore no point of the imaginary axis belongs to K_c.

Solution to Exercise 3.5

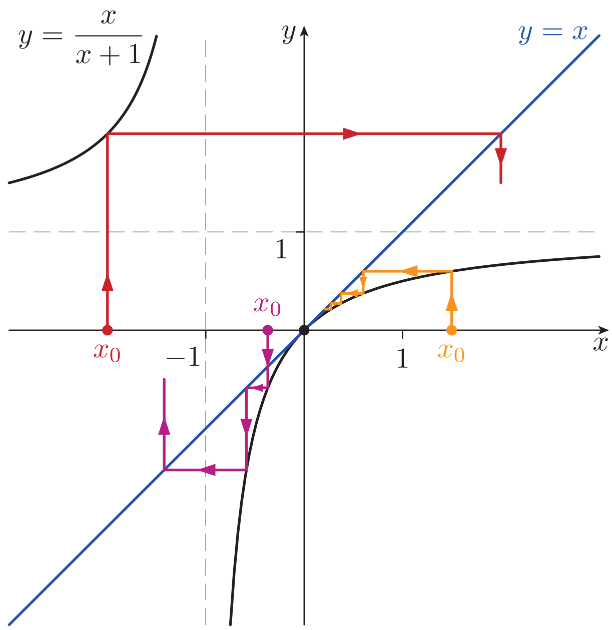


The iteration sequence (x_n) converges to the fixed point $\frac{1}{2}$ of $P_{1/4}$, which is indifferent.

Solution to Exercise 3.6

Let $f(x) = x/(x+1)$, so the sequence (x_n) is obtained by iterating the function f with initial term x_0.

First we plot $y = x$ and $y = f(x)$ on the same diagram, and then we apply graphical iteration with various initial terms x_0.



Since $y = x/(x+1)$ lies below $y = x$ and above the x-axis, for $x > 0$, graphical iteration shows that if $x_0 > 0$, then $x_n \to 0$ as $n \to \infty$.

If $x_0 < -1$, then $x_1 = \dfrac{x_0}{x_0+1} > 0$ and so again $x_n \to 0$ as $n \to \infty$.

If $x_0 = -1$, then $x_1 = f(x_0)$ is not defined. Similarly, the sequence (x_n) has only finitely many terms whenever x_0 lies in the set A, since each of these points maps eventually to -1 under iteration of f, by the hint.

If $-1 < x_0 < 0$ and $x_0 \notin A$, then the terms x_n decrease until $x_{n_0} < -1$ for some positive integer n_0, and hence $x_n \to 0$ as $n \to \infty$.

Since (x_n) evidently tends to 0 if $x_0 = 0$, we deduce that $x_n \to 0$ for all initial values x_0 in $\mathbb{R} - A$.

Solution to Exercise 4.1

In Exercise 3.4 we saw that if $c < -2$, then K_c does not meet the imaginary axis. Since K_c has points in both $G_1 = \{z : \operatorname{Re} z > 0\}$ and $G_2 = \{z : \operatorname{Re} z < 0\}$ (for example, the fixed points $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$), we deduce that K_c is disconnected for $c < -2$.

Solution to Exercise 4.2

(a) Suppose that $c \in \left[-2, \frac{1}{4}\right]$. Then, by Theorem 3.2, K_c contains 0. Hence K_c is connected, by Theorem 4.2, so $c \in M$.

(b) Since K_c is disconnected for $c > \frac{1}{4}$ and $c < -2$, by Exercise 4.1 and the discussion preceding it, we have $c \notin M$ for $c > \frac{1}{4}$ and $c < -2$. Hence, by part (a), $M \cap \mathbb{R} = \left[-2, \frac{1}{4}\right]$.

Solution to Exercise 4.3

(a) If $c = -2$, then the terms of $(P_c^n(0))$ are

$$-2, 2, 2, \dots.$$

Since all these terms lie in $\{z : |z| \le 2\}$, we deduce from Theorem 4.3 that $-2 \in M$.

(b) If $c = 1 + i$, then the terms of $(P_c^n(0))$ are

$$1 + i,\ (1+i)^2 + 1 + i = 1 + 3i,\ \dots.$$

Since $|1 + 3i| = \sqrt{10} > 2$, we deduce from Theorem 4.3 that $1 + i \notin M$.

(c) If $c = i$, then the terms of $(P_c^n(0))$ are

$$i,\ -1+i,\ -i,\ -1+i,\ \dots.$$

Since all these terms lie in $\{z : |z| \le 2\}$, we deduce from Theorem 4.3 that $i \in M$.

(d) If $c = i\sqrt{2}$, then the terms of $(P_c^n(0))$ are

$$i\sqrt{2},\ -2 + i\sqrt{2},\ \dots.$$

Since $|-2 + i\sqrt{2}| = \sqrt{6} > 2$, we deduce from Theorem 4.3 that $i\sqrt{2} \notin M$.

Solution to Exercise 4.4

(a) The point $c = -0.9 + 0.1i$ appears to lie in the disc $|z + 1| < \frac{1}{4}$, so we use Theorem 4.6(b). Since

$$|c + 1| = |0.1 + 0.1i| = 0.1414\ldots < \tfrac{1}{4},$$

P_c has an attracting 2-cycle, by Theorem 4.6(b). Thus c lies in M, by Theorem 4.5.

(b) The point $c = 0.2 + 0.5i$ appears to lie inside the cardioid, so we use Theorem 4.6(a). Since $|c|^2 = 0.29$ and $\operatorname{Re} c = 0.2$, we have

$$\begin{aligned}\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\operatorname{Re} c &= (2.32 - 1.5)^2 + 1.6\\ &= 2.2724 < 3,\end{aligned}$$

so P_c has an attracting fixed point, by Theorem 4.6(a). Thus c lies in M, by Theorem 4.5.

Solution to Exercise 4.5

By Lemma 4.1, if $c \ne -\frac{3}{4}$, then P_c has a single 2-cycle α_1, α_2, where

$$\alpha_1 = -\tfrac{1}{2} + \sqrt{-\tfrac{3}{4} - c}, \quad \alpha_2 = -\tfrac{1}{2} - \sqrt{-\tfrac{3}{4} - c},$$

with multiplier $4(c+1)$.

Hence the 2-cycle is attracting if and only if

$$4|c+1| < 1,$$

that is, if and only if $|c + 1| < \frac{1}{4}$.

Solution to Exercise 4.6

By Theorem 4.7, the function P_c has a super-attracting 3-cycle if and only if

$$P_c^3(0) = (c^2 + c)^2 + c = 0,$$

but $P_c(0) = c \ne 0$ and $P_c^2(0) = c^2 + c \ne 0$.

Thus we seek the solutions of

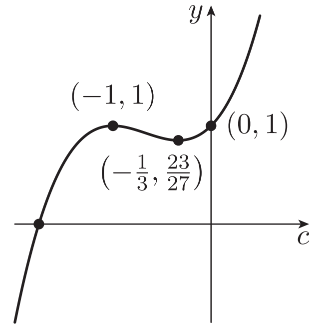
$$c^4 + 2c^3 + c^2 + c = 0$$

which are not $c = 0$ or $c = -1$.

Since $c = 0$ is a solution and $c = -1$ is not, we need to solve

$$c^3 + 2c^2 + c + 1 = 0.$$

Using calculus, we see that the real function $f(c) = c^3 + 2c^2 + c + 1$ has the following graph, so there is only one real zero.



$y = c^3 + 2c^2 + c + 1$

Since $f(-1.8) = -0.152$ and $f(-1.7) = 0.167$, it follows that this value of c lies in $[-1.8, -1.7]$.

The remaining two solutions form a pair of complex conjugates, since f is a real polynomial function.

Remark: The three solutions, correct to three decimal places, are -1.755 and $-0.123 \pm 0.745i$; these solutions can be obtained by using formula (0.6) in the Introduction to Unit A1. See Figure 2.18(b) for the Julia set of P_c in the case $c = -0.123 + 0.745i$.

Solution to Exercise 4.7

(a) Since any solution of $P_c(z) - z = 0$ is also a solution of $P_c^3(z) - z = 0$, we expect $P_c(z) - z$ to be a factor of $P_c^3(z) - z = 0$. Using the formula for $P_c^3(z)$ obtained in Exercise 2.2(a), we find that

$$\begin{aligned} P_c^3(z) - z &= (P_c(z) - z)Q_c(z) \\ &= (z^2 - z + c)Q_c(z), \end{aligned}$$

where

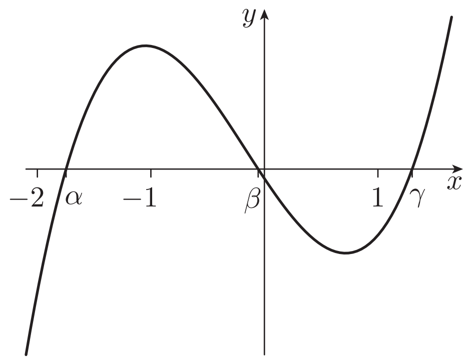
$$\begin{aligned} Q_c(z) = z^6 + z^5 &+ (3c+1)z^4 + (2c+1)z^3 \\ &+ (3c^2 + 3c + 1)z^2 + (c^2 + 2c + 1)z \\ &+ c^3 + 2c^2 + c + 1. \end{aligned}$$

(Begin by finding the coefficient of z^6 and the constant term, and then work inwards from both ends, finding one coefficient at a time and obtaining the coefficient of z^3 twice, as a check.)

(b) On substituting $c = -\frac{7}{4}$ we obtain, after some arithmetic,

$$\begin{aligned} &Q_{-7/4}(z) \\ &= z^6 + z^5 - \tfrac{17}{4}z^4 - \tfrac{5}{2}z^3 + \tfrac{79}{16}z^2 + \tfrac{9}{16}z + \tfrac{1}{64} \\ &= \left(z^3 + \tfrac{1}{2}z^2 - \tfrac{9}{4}z - \tfrac{1}{8}\right)^2. \end{aligned}$$

(c) Using calculus, we see that the graph of $y = x^3 + \frac{1}{2}x^2 - \frac{9}{4}x - \frac{1}{8}$ is as follows.



$y = x^3 + \frac{1}{2}x^2 - \frac{9}{4}x - \frac{1}{8}$

Hence the equation $x^3 + \frac{1}{2}x^2 - \frac{9}{4}x - \frac{1}{8} = 0$ has three real solutions α, β, γ, say. Therefore

$$z^3 + \tfrac{1}{2}z^2 - \tfrac{9}{4}z - \tfrac{1}{8} = (z-\alpha)(z-\beta)(z-\gamma). \quad \text{(S3)}$$

By part (b), α, β, γ are the only solutions of $P^3_{-7/4}(z) - z = 0$ that are not fixed points of $P_{-7/4}$, so they must form a 3-cycle of $P_{-7/4}$.

(d) By Theorem 2.4(a), the multiplier of the 3-cycle is

$$\begin{aligned} (P_c^3)'(\alpha) &= P_c'(\alpha)P_c'(\beta)P_c'(\gamma) \\ &= (2\alpha)(2\beta)(2\gamma) \\ &= 8\alpha\beta\gamma \\ &= 1, \end{aligned}$$

since $\alpha\beta\gamma = \frac{1}{8}$, by equation (S3).

Hence, by Theorem 4.8(a), a saddle-node bifurcation occurs at $c = -7/4$, so we expect to see a small cardioid-shaped periodic region with cusp at this point. This is included in Figure 4.9, and its centre is at the point $c = -1.755$, correct to three decimal places, which is one of the three points found in the solution to Exercise 4.6.

Solution to Exercise 4.8

Since

$$P_c(\zeta) = \zeta^2 + c = \zeta^2 + (\zeta - \zeta^2) = \zeta,$$

we find that ζ is a fixed point of P_c. Also, the multiplier is

$$P_c'(\zeta) = 2\zeta,$$

which is a root of unity ($\neq 1$). Hence, by Theorem 4.8(b), a period-multiplying bifurcation occurs at c. (In fact, since the main cardioid is the image of the circle $|z| = \frac{1}{2}$ under the function $f(z) = z - z^2$, the point c lies on the cardioid.)

If $\zeta = -\frac{1}{2}$, then $2\zeta = -1$, which is a primitive square root of unity, so a period-doubling bifurcation occurs at

$$c = -\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)^2 = -\tfrac{3}{4}.$$

This is the point where the two periodic regions in Theorem 4.6 touch, which is visible in Figure 4.9.

If $\zeta = \frac{1}{2}e^{2\pi i/3}$, then $2\zeta = e^{2\pi i/3}$, which is a primitive cube root of unity, so a 'period-trebling' bifurcation occurs at

$$c = \tfrac{1}{2}e^{2\pi i/3} - \left(\tfrac{1}{2}e^{2\pi i/3}\right)^2 = -\tfrac{1}{8} + \tfrac{3}{8}\sqrt{3}i,$$

and this is visible in Figure 4.9.

If $\zeta = \frac{1}{2}i$, then $2\zeta = i$, which is a primitive fourth root of unity, so a 'period-quadrupling' bifurcation occurs at

$$c = \tfrac{1}{2}i - \left(\tfrac{1}{2}i\right)^2 = \tfrac{1}{4} + \tfrac{1}{2}i,$$

and this is also visible in Figure 4.9.

Solution to Exercise 4.9

(a) Let $c = -1 + 2i$. Since $|P_c(0)| = |c| = \sqrt{5} > 2$, we deduce from Theorem 4.3 that $c \notin M$.

(b) Since $P_{-1}(0) = -1$ and $P_{-1}(-1) = 0$, we deduce that the sequence $(P_{-1}^n(0))$ is $0, -1, 0, -1, \ldots$. Hence

$$\left|P_{-1}^n(0)\right| \leq 2, \quad \text{for } n = 1, 2, \ldots,$$

so $-1 \in M$, by Theorem 4.3.

(c) For $c = -1 + i$, we have

$$\begin{aligned}
P_c(0) &= -1 + i,\\
P_c^2(0) &= (-1+i)^2 + (-1+i) = -1 - i,\\
P_c^3(0) &= (-1-i)^2 + (-1+i) = -1 + 3i.
\end{aligned}$$

Since

$$|P_c^3(0)| = \sqrt{10} > 2,$$

we deduce from Theorem 4.3 that $c \notin M$.

Solution to Exercise 4.10

Suppose that $|c| = 2$ but $c \neq -2$. Then

$$|c^2 + c| = |c(c+1)| = |c|\,|c+1| = 2|c+1|.$$

Now, since $|c| = 2$, we have

$$\begin{aligned}
|c+1|^2 &= (c+1)\overline{(c+1)}\\
&= |c|^2 + 2\operatorname{Re} c + 1\\
&= 5 + 2\operatorname{Re} c.
\end{aligned}$$

We know that $|c| = 2$ but $c \neq -2$, so it follows that $\operatorname{Re} c > -2$. Hence

$$|c+1|^2 = 5 + 2\operatorname{Re} c > 5 - 2 \times 2 = 1.$$

Therefore $|c+1| > 1$, so

$$|P_c^2(0)| = |c^2 + c| = 2|c+1| > 2.$$

Hence $c \notin M$, by Theorem 4.3.

Solution to Exercise 4.11

(a) Let $c = -1.1 - 0.1i$. Then

$$|c+1| = |-0.1 - 0.1i| = 0.1414\ldots < \tfrac{1}{4}.$$

Hence, by Theorem 4.6(b), P_c has an attracting 2-cycle and so, by Theorem 4.5, $c \in M$.

(b) Let $c = 0.6i$. Then $|c|^2 = 0.36$ and $\operatorname{Re} c = 0$, so

$$\begin{aligned}
\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\operatorname{Re} c &= (8 \times 0.36 - 1.5)^2 + 0\\
&= 1.9044 < 3.
\end{aligned}$$

Hence, by Theorem 4.6(a), P_c has an attracting fixed point and so, by Theorem 4.5, $c \in M$.

Solution to Exercise 4.12

(a) If some point $\alpha \in \partial\mathcal{R}$ lies outside M, then because M is closed there is an open disc D with centre α that lies entirely outside M. Since $\mathcal{R} \subseteq M$, the open disc D does not meet $\mathcal{R}$, and this contradicts the fact that α is a boundary point of $\mathcal{R}$. Hence $\partial\mathcal{R} \subseteq M$.

(b) (i) From Figure 4.9, it appears that the point $c = \frac{1}{4} + \frac{1}{2}i$ lies on the main cardioid. This can be verified by checking that c satisfies

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c = 3.$$

Thus c lies on the boundary of the periodic region in M given by Theorem 4.6(a), and hence $c \in M$ by part (a).

(ii) The point $c = -1 + \frac{1}{4}i$ lies on the circle $|c+1| = \frac{1}{4}$. Thus c lies on the boundary of the periodic region in M given by Theorem 4.6(b), and hence $c \in M$ by part (a).

Solution to Exercise 4.13

By Theorem 4.7, the function P_c has a super-attracting 4-cycle if and only if $P_c^4(0) = 0$, but $P_c(0)$, $P_c^2(0)$ and $P_c^3(0)$ are non-zero.

Now

$$\begin{aligned} P_c(0) &= c, \\ P_c^2(0) &= c^2 + c, \\ P_c^3(0) &= c^4 + 2c^3 + c^2 + c, \end{aligned}$$

and $P_c^4(0)$ is of the form

$$P_c^4(0) = c^8 + 4c^7 + \cdots + c^2 + c.$$

If $P_c(0) = 0$ or $P_c^2(0) = 0$, then

$$P_c^4(0) = P_c^2(P_c^2(0)) = 0.$$

Hence any zero of $P_c^2(0)$ must also be a zero of $P_c^4(0)$, so $P_c^4(0)$ must have $P_c^2(0) = c^2 + c$ as a factor:

$$P_c^4(0) = (c^2 + c)(c^6 + 3c^5 + \cdots + 1).$$

Thus the values of c for which P_c has a super-attracting 4-cycle must be solutions of the equation

$$c^6 + 3c^5 + \cdots + 1 = 0,$$

and there are at most six such (distinct) solutions. Hence there are at most six values of c for which P_c has a super-attracting 4-cycle.

Remark: In fact it can be shown that each of the polynomial functions $c \longmapsto P_c^n(0)$ has only simple zeros. This makes it possible to count the number of values of c for which P_c has a super-attracting p-cycle, for each positive integer p. In particular, there are exactly six values of c for which P_c has a super-attracting 4-cycle, as indicated in Figure 4.9.

Solution to Exercise 4.14

For $c = -\frac{5}{4}$, P_c has the 2-cycle

$$\alpha_1 = -\tfrac{1}{2} + \sqrt{\tfrac{1}{2}}, \quad \alpha_2 = -\tfrac{1}{2} - \sqrt{\tfrac{1}{2}}$$

(by Lemma 4.1) with multiplier

$$4(c+1) = 4\left(-\tfrac{5}{4} + 1\right) = -1.$$

Since -1 is a primitive square root of unity, we deduce, by Theorem 4.8(b), that a period-doubling bifurcation occurs at $c = -\frac{5}{4}$.

In Figure 4.9 we see that the point $-\frac{5}{4}$ lies where the periodic region with period 2 (the disc $\left\{c : |c+1| < \frac{1}{4}\right\}$) and a periodic region with period 4 touch, so a period-doubling bifurcation is visible there.

References

Douady, A. (1986) ‘Julia sets and the Mandelbrot set’, in Peitgen, H. O. and Richter, P. H. (eds) *The Beauty of Fractals: Images of Complex Dynamical Systems*, Berlin, Springer.

Acknowledgements

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Unit D1

Nikolay Yegorovich Zhukovsky (end of Section 3): SPUTNIK / Alamy Stock Photo

Figure 5.6: ONERA photograph, Werle 1974

Unit D2

Gaston Julia (end of Section 2): original source unknown

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